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Journal of Geometry and Physics 32 (1999) 14–56

JOURNAL OF GEOMETRY AND PHYSICS

Invariant local twistor calculus for quaternionic structures and related geometries

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Received 6 May 1998

Abstract

New universal invariant operators are introduced in a class of geometries which include the quaternionic structures and their generalizations as well as four-dimensional conformal (spin) geometries. It is shown that, in a broad sense, all invariants and invariant operators arise from these universal operators and that they may be used to reduce all invariants problems to corresponding algebraic problems involving homomorphisms between modules of certain parabolic subgroups of Lie groups. Explicit application of the operators is illustrated by the construction of all non-standard operators between exterior forms on a large class of the geometries which includes the quaternionic structures. © 1999 Published by Elsevier Science B.V. All right reserved.

Subj. Class.: Spinors and twistors
1991 MSC: 32L25; 53A50; 53A55; 53C10; 53C15
Keywords: Twistor calculus; Conformal spin manifolds; Quaternionic manifolds; Almost Grassmannian manifolds; Invariant operators

1. Introduction

A *real almost Grassmannian structure* on a manifold M (briefly a *real AG-structure*) is given by a fixed identification of the tangent bundle TM with the tensor product of two auxiliary vector bundles of dimensions p and q, together with the identification of their top

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[1] Australian Research Council QEII Research Fellow.

[2] Supported by Australian Research Council, University of Adelaide, and GACR grant no. 201/96/0310.

PII: S0393-0440(99)00018-2

degree exterior powers. In the realm of Penrose's abstract index notation, we shall express this by

$$\mathcal{E}^a = \mathcal{E}_{A'} \otimes \mathcal{E}^A = \mathcal{E}^A_{A'}, \quad \wedge^q \mathcal{E}^A \simeq \wedge^p \mathcal{E}_{A'}. \tag{1}$$

Equivalently, this amounts to the reduction of the structure group $\mathrm{GL}(pq, \mathbb{R})$ of the tangent bundle to its subgroup $G_0 = \mathrm{S}(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R}))$. Thus the complexified tangent bundle of a real AG-structure is equipped by the reduction of its structure group to $G_0^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$. There is another class of geometries on $4m$-dimensional manifolds with similar behaviour. The geometries are defined by reductions of the structure groups of the tangent bundles to the groups $G_0 = \mathrm{S}(\mathrm{GL}(p/2, \mathbb{H}) \times \mathrm{GL}(q/2, \mathbb{H})) \subset \mathrm{GL}(pq, \mathbb{R})$ with $2 \le p \le q$ even, and the complexifications of their tangent bundles enjoy again the fundamental identification (1). The most important algebraic feature of the two types of the structures above is that, for each pair p, q, their respective structure groups G_0 are the maximal reductive parts of certain maximal parabolic subgroups P in two different real forms G of the same complex semi-simple group $G^{\mathbb{C}} = \mathrm{SL}(p+q, \mathbb{C})$. A geometry will be called an *AG-structure* if it has a structure group G_0 where G_0 is a maximal reductive part of a parabolic $P \subset G$ such that $P = G \cap P^{\mathbb{C}}$ with $G^{\mathbb{C}} = \mathrm{SL}(p+q, \mathbb{C})$ and where $P^{\mathbb{C}}$ is the maximal parabolic in $G^{\mathbb{C}}$ such that $G^{\mathbb{C}}/P^{\mathbb{C}}$ is the Grassmannian of complex p-planes in $\mathbb{C}^{p+q}$ (with $2 \le p \le q$). Thus the members of the list of all AG-structures are named by such pairs (G, P) and in fact the real group G is one of the following: $G = \mathrm{SL}(p+q, \mathbb{R})$ with $2 \le p \le q$, $G = \mathrm{SL}(p/2 + q/2, \mathbb{H})$ and p, q are even, or $G = \mathrm{SU}(p, p)$, see Appendix A for more details. Henceforth G and P will indicate such a pair and G_0 will be the reductive part of P. The identification (1) of the complexified tangent spaces is given for all the AG-structures. The complex almost Grassmannian structures on complex manifolds were studied in [1] under the name 'paraconformal manifolds'. Similar objects were introduced earlier in [12], see also [11].

The most well-known examples of such structures are four-dimensional conformal spin structures (here $G_0 = \mathbb{R} \cdot \mathrm{Spin}(p, q, \mathbb{R}) \subset \mathrm{Spin}(p+1, q+1, \mathbb{R})$, $p + q = 4$, and the complexification $\mathrm{Spin}(6, \mathbb{C}) \simeq \mathrm{SL}(4, \mathbb{C})$). We will extend the term 'spinor' from that case and in all cases deem the auxiliary bundles $\mathcal{E}_{A'}$ and $\mathcal{E}^A$ to be *spinor bundles*.

The almost quaternionic structures on manifolds are classical 1st order G-structures, such that their structure group G_0 is the subgroup $\mathrm{GL}(m, \mathbb{H}) \times_{\mathbb{Z}_2} \mathrm{Sp}(1) \subset \mathrm{GL}(4m, \mathbb{R})$, see [22]. We have to notice that the action of G_0 on $\mathbb{H}^m$ (i.e. the indicated embedding into the real general linear group) is defined by the adjoint action of the block-diagonal matrices in $\mathrm{GL}(1+m, \mathbb{H})$ on the block below the diagonal. The group $\tilde{G}_0 = S(\mathrm{GL}(1, \mathbb{H}) \times \mathrm{GL}(m, \mathbb{H}))$ is the universal cover of G_0 and the choice of the structure group $\tilde{G}_0$ makes no difference locally. In particular, the almost quaternionic structures belong to our class of AG-structures. They are called *quaternionic* if they admit a torsion-free connection. It was pointed out in [22], and worked out in much detail in [1,3], that these structures fit into a larger class of geometries coming from the so-called |1|-graded semi-simple Lie algebras. This is exactly our point of view and the corresponding entry in our list of pairs (G, P) is that with $G = \mathrm{SL}(1 + q/2, \mathbb{H})$, $q \ge 2$ even.

Despite the very transparent geometric differences between, for example, the real almost Grassmannian structures and the almost quaternionic structures, we will treat all these cases simultaneously. In the cases corresponding to the 'split real form' $G = \mathrm{SL}(p+q, \mathbb{R})$ we will write TM to mean the usual tangent bundle while for the other cases TM will mean the complexification of the tangent bundle. Similarly for P-modules, and the bundles they induce, we will take these to be real for the geometries of the split real forms but complex for the geometries corresponding to the other groups. With this understood we will suppress explicit reference to the scalars concerned and write, for example, $\mathrm{SL}(m)$ for either the real or complex special linear group as required by context. These conventions will enable us to use the same index formalism for all these geometries and also enable us to avoid complexifying except where necessary.

Treating all such AG-structures simultaneously, the main results we obtain are as follows:

- We construct a new invariant first order differential operator that we call a twistor-D operator – see Definition 3.1. This operator may be viewed as an analogue, for these structures, of the Levi-Civita connection of Riemannian geometry.
- Via the twistor-D operator we construct curved analogues of all the non-standard operators between exterior differential forms on a class of AG-structures that includes all the quaternionic geometries – see Theorem 5.1.
- We use the twistor-D operator to construct a module for an appropriate parabolic subgroup P such that all invariant differential operators (linear and polynomial and up to any chosen order) and invariants of AG-structures are equivalent to P-homomorphisms from this module to irreducible P-modules. See in particular Theorem 4.4. The implications of this are discussed below.

We should also point out that considerable detail of a 'calculus' to enable manipulation and application of the twistor-D operator and its accompanying machinery is presented. Most of this is strictly needed to establish the results mentioned. However, we have attempted to present this in an explicit form that could be directly used by readers as we believe that there are many potential applications of these tools in mathematical-physics, especially since they include new tools for the four-dimensional conformal structures and their associated twistor theory. For example the twistor-D operator should be particularly useful for the construction and study of conformally invariant spinor equations in four-dimensions. In addition to the main results there are other observations and results along the way. In particular, in Section 6 we observe an obvious extension to Salamon's complex, we relate the twistor-D operator to the so-called tractor-D operator of conformal geometry and we also generalise the latter to a class of AG-structures.

Underlying our constructions here is the result that a manifold with an almost Grassmannian structure comes equipped with a canonical *Cartan bundle* $\mathcal{G} \to M$ and associated canonical Cartan connection. In each case $\mathcal{G}$ is a principal fibre bundle with structure group P where this is a maximal parabolic subgroup of a Lie group G as above. The canonical *(normal) Cartan connection* ω is a special 1-form on $\mathcal{G}$ which takes values in the Lie algebra $\mathfrak{g}$ of G and gives a complete parallelization of $\mathcal{G}$ (see Appendix A and [3,7] for more details). The Cartan bundle may be regarded as a deformation of the homogeneous situation where one has G as principal bundle with fibre P over G/P and in this latter picture the Cartan connection reduces to the Maurer Cartan form. As in the homogeneous case each P-module V gives rise to an induced or *natural bundle* $\mathcal{V}$. Moreover, in the special case of a G-module W the corresponding natural bundle $\mathcal{W}$ comes equipped with a canonical linear connection (also denoted by ω). Such bundles will be described here by what will be called *(local) twistor bundles* and their canonical connections will be viewed as *twistor connections*. Since, in the current work, we are concerned with the production of explicit operators on M we avoid a detailed discussion of the Cartan bundle and work directly on these induced natural bundles and their connections. Indeed most of the work can be understood without a deep understanding of the inducing Cartan bundles. However, we would like to point out that many of the 'background results' can be recovered most efficiently from the principal bundle point of view and Appendix A is dedicated to extracting from the general theory of Cartan bundles and their connections (as in for example [6]) the results required for the current work.

Calculus similar to the twistor calculus we develop here has been successfully applied to other related geometries. For example in [13] a first order invariant tractor-D operator (rediscovered in [2] but originally due to Tracy Thomas), and some calculus based around this, is used to construct all density valued invariants of projective geometries. In [14] a similar programme is in place to produce a complete invariant theory for conformal geometries and there are already many new results in this. Such calculus has also been used to proliferate invariant operators on conformal, projective and CR structures. As with the AG-structures studied here, these geometries are all 'parabolic geometries' which may be viewed as deformations of homogeneous structures G/P where G is semi-simple and P a parabolic subgroup. It turns out that at each point of such a structure P acts on the jet information (jets of the geometric structure itself or jets of a field on the structure). Understanding and dealing with this action is the key problem. This is difficult and subtle in general and many papers have discussed similar problems, see e.g. [3,6,8,24], and the references therein. Roughly speaking the programme here, as with the tractor calculus, is to use the twistor-D operator to package this jet information into 'parcels' which are P-submodules of irreducible G-modules. This is a huge step since at least the latter G-modules are understood and can be dealt with by classical techniques such as Weyl's invariant theory. (A discussion of the general programme, in the context of tractor calculus, as well as other results are described in [15].) Then invariants and invariant operators may be proliferated by identifying the relevant P-submodules of irreducible G-modules. That all invariants and invariant operators are equivalent to the corresponding P-homomorphisms is the content of Theorem 4.4. A more intuitive interpretation of this result is that all invariants arise from the twistor-D operator (and its concatenations – the universal invariant $\mathsf{D}^{(k)}$ operators of Section 3). As far as we know this is the first theorem of its sort and thus far there is no corresponding theorem established for the tractor operators. This theorem leaves open the question of whether the remaining P-submodule problems are tractable. Evidence that in many important cases they are is the success of the analogous tractor calculus, as mentioned above, and more importantly for this case the application of the twistor-D operator to produce the new family of invariant differential operators in Section 5. For future work in this direction, as well as to develop some results needed here, Appendix C discusses the composition series of submodules in a rather general setting.

The plan of the paper goes as follows. After setting notation and outlining some further preliminaries in Section 2, we introduce the twistor-D operators. Section 4 is devoted to the main theorem, Theorem 4.4, the proof of which relies on an explicit description of the normal forms of the AG-structures, cf. Appendix B. Then we proceed with our main application, the curved analogues of the non-standard operators on exterior forms. These are fourth order and include analogies to the square of the Laplacian in four-dimensional conformal geometries. Further observations, as mentioned above, in Section 6, are followed by the three appendices.

2. Preliminaries

Here we review some important technicalities and introduce our notational conventions. We omit explicit verifications of most of the claims as they follow easily from the general theory as reviewed in Appendix A, see also [6]. For an explicit development (although in the complex setting) with notation and conventions very similar to those here see [1].

Index formalism. Except where otherwise indicated we use Penrose's abstract index notation [20] which allows for easy explicit calculations without involving a choice of basis. Thus we may write, v^A or v^B for a section of the unprimed fundamental spinor bundle $\mathcal{E}^A$. Similarly $w_{A'}$ could denote a section of the primed fundamental spinor bundle $\mathcal{E}_{A'}$. We write $\mathcal{E}_A$ for the dual bundle to $\mathcal{E}^A$ and $\mathcal{E}^{A'}$ for the dual to $\mathcal{E}_{A'}$. The tensor products of these bundles yield the general spinor objects such as $\mathcal{E}_{AB} := \mathcal{E}_A \otimes \mathcal{E}_B$, $\mathcal{E}^{ABC'}_{A'B'}$ and so forth. The tensorial indices are also abstract indices. Recall (see above) that $\mathcal{E}^a = \mathcal{E}^A_{A'}$ is the tangent bundle, so $\mathcal{E}_a = \mathcal{E}^{A'}_A$ is the cotangent bundle and we may use the terms 'spinor' or 'section of a spinor bundle' to describe tensor fields.

A spinor object on which some indices have been contracted will be termed a *contraction* (of the underlying spinor). For example

$$v^{ABC'}_{BC'DE}$$

is a contraction of $v^{ABC'}_{DD'EF}$. In many cases the underlying spinor of interest is a tensor product of lower valence spinors. For example

$$v^{AB} w^{C'}_B u_{ACD}$$

is a contraction of $v^{AB} w^{C'}_C u_{DEF}$. The same conventions are used for the tensor indices and the twistor indices; the latter are introduced below. Standard notation is also used for the symmetrizations and antisymmetrizations over some indices.

Weights and scales. We define line bundles of densities or *weighted functions* as follows. The weight -1 line bundle $\mathcal{E}[-1]$ over M is identified with

$$\mathcal{E}^{\overbrace{[A'B'\cdots C']}^{p}}.$$

Then, for integral w, the weight w line bundle $\mathcal{E}[w]$ is defined to be $(\mathcal{E}[-1])^{-w}$. In fact in the case of AG-geometries corresponding to the real-split form $\mathrm{SL}(p+q,\mathbb{R})$ we can (locally) extend this definition to weights $w \in \mathbb{R}$ by locally selecting a ray fibre subbundle of $\mathcal{E}[-1]$. Calling this say $\mathcal{E}_+[-1]$ we can then define the ray bundles $\mathcal{E}_+[w] := (\mathcal{E}[-1])^{-w}$. Finally these may be canonically extended to line bundles in the obvious way. In any case we write $\mathcal{E}^{A'}[w]$ for $\mathcal{E}^{A'} \otimes \mathcal{E}[w]$ and so on, whenever defined. In view of the defining isomorphism

$$h : \wedge^q \mathcal{E}^A \xrightarrow{\cong} \wedge^p \mathcal{E}_{A'} \tag{2}$$

we also have

$$\mathcal{E}[-1] \cong \mathcal{E}_{\underbrace{[AB\cdots C]}_{q}}, \quad \mathcal{E}[1] \cong \mathcal{E}^{\overbrace{[AB\cdots C]}^{q}} \cong \mathcal{E}_{\underbrace{[A'\cdots C']}_{p}}.$$

We write $\epsilon^{A'B'\cdots C'}$ for the tautological section of $\mathcal{E}^{[A'B'\cdots C']}[1]$ giving the mapping $\mathcal{E}[-1] \xrightarrow{\cong} \wedge^p \mathcal{E}^{A'}$ by

$$f \mapsto f\epsilon^{A'B'\cdots C'}, \tag{3}$$

and $\epsilon_{D\cdots E}$ for similar object giving $\mathcal{E}[-1] \xrightarrow{\cong} \wedge^q \mathcal{E}_A$. A *scale* for the AG-structure is a nowhere vanishing section ξ of $\mathcal{E}[1]$. Note that such a choice is equivalent to a choice of spinor 'volume' form

$$\epsilon_\xi^{A'\cdots C'} := \xi^{-1}\epsilon^{A'\cdots C'},$$

or to a choice of form,

$$\epsilon^\xi_{D\cdots E} := \xi^{-1}\epsilon_{D\cdots E}.$$

Distinguished connections. A connection ∇_a on M belongs to the given AG-structure (this really means ∇_a comes from a principal connection on the bundle $\mathcal{G}_0$ described below) if and only if it satisfies two conditions:

- ∇_a is the tensor product of linear connections (both of which we shall also denote ∇_a) on the spinor bundles $\mathcal{E}^A$ and $\mathcal{E}_{A'}$,
- the defining isomorphism h in (2) is covariantly constant, i.e. $\nabla_a h = 0$.

Our conventions for the torsion $T_{ab}{}^c$ and curvature $R_{ab}{}^c{}_d$ of a connection ∇_a on the tangent bundle TM are determined by the following equation:

$$2\nabla_{[a}\nabla_{b]}v^c = T_{ab}{}^d\nabla_d v^c + R_{ab}{}^c{}_d v^d.$$

Since $T_{ab}{}^c$ is skew on its lower indices, $T_{ab}{}^c = T_{[ab]}{}^c$, it can be written as a sum of two terms

$$T_{ab}{}^c = F_{ab}{}^c + \tilde{F}_{ab}{}^c,$$

where

$$F_{ab}{}^c =: F_{ABC'}^{A'B'C} = F_{(AB)C'}^{[A'B']C} \quad \text{and} \quad \tilde{F}_{ab}{}^c =: \tilde{F}_{ABC'}^{A'B'C} = F_{[AB]C'}^{(A'B')C}.$$

The Cartan bundle $\mathcal{G}$ over the manifold M has the quotient $\mathcal{G}_0$, a principal fibre bundle with structure group G_0. By the general theory, each G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the quotient projection defines the distinguished principal connection on $\mathcal{G}_0$, the pullback of the $\mathfrak{g}_0$-part of ω. The whole class of these connections consists precisely of connections on $\mathcal{G}_0$ with the unique torsion taking values in the kernel of ∂^*. A straightforward computation shows that the latter condition is equivalent to the condition that both $\tilde{F}$ and F be completely trace-free (cf. Appendix A). Each principal connection on $\mathcal{G}_0$ induces the induced connection on the bundle $\mathcal{E}[1]\setminus\{0\}$ which is associated to $\mathcal{G}_0$ and, moreover, the resulting correspondence between the sections σ and the latter connections is bijective. In particular, each section ξ of the bundle $\mathcal{E}[1]\setminus\{0\}$ defines uniquely a reduction σ, such that the corresponding distinguished connection leaves ξ horizontal. Altogether we have recovered Theorems 2.2 and 2.4 of [1]. We rephrase these here for convenience:

Theorem 2.1. *Given a scale ξ on an AG-structure there are unique connections on $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ such that $F_{ABC'}^{A'B'C}$ and $\tilde{F}_{ABC'}^{A'B'C}$ are totally trace-free, the induced covariant derivative preserves the isomorphism h of (2), and $\nabla_a\xi = 0$. The torsion components $F_{ab}{}^c$ and $\tilde{F}_{ab}{}^c$ of the induced connection on TM are invariants of the AG-structures.*

Notice that in the special case of the four-dimensional conformal geometries, there is always a connection with vanishing torsion on $\mathcal{G}_0$ and so both F and $\tilde{F}$ are zero. The scales correspond to a choice of metric from the conformal class while the general distinguished connections (corresponding to the reduction parameter σ being not necessarily exact) are just the Weyl geometries.

We may write ∇_a^{ξ} to indicate a connection as determined by the theorem, although mostly we will omit the ξ. Thus we might write $\nabla_a^{\hat{\xi}}$ or simply $\hat{\nabla}_a$ to indicate the connection corresponding to a scale $\hat{\xi}$ and similar conventions will be used for other operators and tensors that depend on ξ.

In what follows, for the purpose of explicit calculations, we shall often choose a scale and work with the corresponding connections. Objects are then well-defined, or *invariant*, on the AG-structure if they are independent of the choice of scale. Note that if we change the scale according to $\xi \mapsto \hat{\xi} = \Omega^{-1}\xi$, where Ω is a smooth non-vanishing function, then the connection transforms as follows:

$$\begin{aligned}
\mathcal{E}^A &: \hat{\nabla}_A^{A'} u^C = \nabla_A^{A'} u^C + \delta_A^C \Upsilon_B^{A'} u^B, \\
\mathcal{E}_{A'} &: \hat{\nabla}_A^{A'} u_{C'} = \nabla_A^{A'} u_{C'} + \delta_{C'}^{A'} \Upsilon_A^{B'} u_{B'}, \\
\mathcal{E}_B &: \hat{\nabla}_A^{A'} v_B = \nabla_A^{A'} v_B - \Upsilon_B^{A'} v_A, \\
\mathcal{E}^{B'} &: \hat{\nabla}_A^{A'} v^{B'} = \nabla_A^{A'} v^{B'} - \Upsilon_A^{B'} v^{A'},
\end{aligned} \tag{4}$$

where $\Upsilon_a := \Omega^{-1}\nabla_a\Omega$. Consequently

$$\widehat{\nabla}_a f = \nabla_a f + w\Upsilon_a f \tag{5}$$

if $f \in \mathcal{E}[w]$. All these formulae follow from the general discussion in Appendix A, but they are also easily checked directly.

Given a choice of scale ξ, we will write $R_{ab}{}^{C}_{D}$ (or $R^{(\xi)}_{ab}{}^{C}_{D}$ to emphasize the choice of scale) for the curvature of ∇_a on $\mathcal{E}^A$ and $R_{ab}{}^{C'}_{D'}$ for the curvature of ∇_a on $\mathcal{E}_{A'}$, i.e.

$$(2\nabla_{[a}\nabla_{b]} - T_{ab}{}^{e}\nabla_e)v^C = R_{ab}{}^{C}_{D}v^D, \quad (2\nabla_{[a}\nabla_{b]} - T_{ab}{}^{e}\nabla_e)w_{D'} = -R_{ab}{}^{C'}_{D'}w_{C'}.$$

Then the curvature of the induced linear connection on TM is

$$R_{ab}{}^{c}_{d} = R_{ab}{}^{C'}_{D'}\delta^C_D + R_{ab}{}^{C}_{D}\delta^{C'}_{D'}.$$

Observe that since ∇_a preserves the volume forms $\epsilon^{\xi}_{A'\cdots C'}$ and $\epsilon^{\xi}_{D\cdots E}$ it follows that $R_{ab}{}^{C}_{D}$ and $R_{ab}{}^{C'}_{D'}$ are trace-free on the spinor indices displayed. Thus the equations

$$R_{ab}{}^{C}_{D} = U_{ab}{}^{C}_{D} - \delta^C_B\mathsf{P}^{A'B'}_{AD} + \delta^C_A\mathsf{P}^{B'A'}_{BD}$$

and

$$R_{ab}{}^{C'}_{D'} = U_{ab}{}^{C'}_{D'} + \delta^{B'}_{D'}\mathsf{P}^{A'C'}_{AB} - \delta^{A'}_{D'}\mathsf{P}^{B'C'}_{BA}$$

determine the objects $U_{ab}{}^{C}_{D}$, $U_{ab}{}^{C'}_{D'}$ and the *Rho-tensor*, P_{ab}, if we require that $U^{A'B'C}_{ACD} = 0 = U^{A'D'C'}_{ABD'}$. In this notation we have

$$R_{ab}{}^{c}_{d} = U_{ab}{}^{c}_{d} + \delta^{D'}_{C'}\delta^C_A\mathsf{P}^{B'A'}_{BD} - \delta^{D'}_{C'}\delta^C_B\mathsf{P}^{A'B'}_{AD} - \delta^C_D\delta^{A'}_{C'}\mathsf{P}^{B'D'}_{BA} + \delta^C_D\delta^{B'}_{C'}\mathsf{P}^{A'D'}_{AB}, \tag{6}$$

where

$$U_{ab}{}^{c}_{d} = U_{ab}{}^{C}_{D}\delta^{D'}_{C'} + U_{ab}{}^{D'}_{C'}\delta^C_D. \tag{7}$$

In the case of $p = 2 = q$ this agrees with the usual decomposition of the curvature of the Levi-Civita connection into the conformally invariant (and trace-free) Weyl tensor part and the remaining part given by the Rho-tensor (see, e.g. [2]). All these equations also follow from the general definitions of the U's and P's in (A.7). Note that U's are 2-forms valued in $\mathfrak{g}_0$ coming from the curvature of the canonical Cartan connection and so they are in the kernel of ∂^*. This is the source of the condition on the trace, but they are not trace-free in general

$$U_{ab}{}^{C}_{C} = -U_{ab}{}^{C'}_{C'} = 2\mathsf{P}_{[ab]}. \tag{8}$$

On the other hand, it follows from the Bianchi identity,

$$R_{[ab}{}^{d}{}_{c]} + \nabla_{[a}T_{bc]}{}^{d} + T_{[ab}{}^{e}T_{c]e}{}^{d} = 0,$$

that

$$2(p+q)\mathsf{P}_{[ab]} = -\nabla_c T_{ab}{}^{c}. \tag{9}$$

The Rho-tensor P_{ab} has the transformation equation

$$\hat{\mathsf{P}}^{A'B'}_{AB} = \mathsf{P}^{A'B'}_{AB} - \nabla^{A'}_{A}\Upsilon^{B'}_{B} + \Upsilon^{B'}_{A}\Upsilon^{A'}_{B}. \tag{10}$$

Again, this can be easily checked directly but we give a general explanation in (A.6).

We are most interested in the special case $p = 2$. Then the whole component $F_{ab}{}^c$ is irreducible and so it vanishes by our condition on the trace, while the other component $\tilde{F}_{ab}{}^c$ of the torsion, together with the trace-free part of $U^{[A'B']D}_{(ABC)}$ are the only local invariants of the structures (i.e. the AG-structure is locally flat if and only if these two vanish). In all other cases $2 < p \leq q$, the two components of the torsion are the only invariants, cf. the end of Appendix A.

The totally symmetrized covariant derivatives of the Rho-tensors will play a special role. We will use the notation

$$S_{a\cdots b} := \underbrace{\nabla_{(a}\nabla_b \cdots \nabla_d \mathsf{P}_{ef)}}_{s}$$

for $s = 2, 3, \ldots$

Twistors. Via the Cartan bundle $\mathcal{G}$ over M any P-module V gives rise to a *natural bundle* (or induced bundle) $\mathcal{V}$. Sections of $\mathcal{V}$ are identified with functions $f : \mathcal{G} \to V$ such that $f(x \cdot p) = \rho(p^{-1}) f(x)$, where $x \mapsto x \cdot p$ gives the action of $p \in P$ on $x \in \mathcal{G}$ while ρ is the action defining the P-module structure.

Recall also that the Cartan bundle is equipped with a canonical connection, the so-called normal Cartan connection ω. In view of this it is in our interests to work, where possible, with natural bundles $\mathcal{V}$ induced from V where this is not merely a P-module but in fact a G-module. Then the Cartan connection induces an invariant linear connection on $\mathcal{V}$. Let us write V^α for the module corresponding to the standard representation of G on $\mathbb{R}^{p+q}$ and write V_α for the dual module. The index α is another Penrose-type abstract index and we write $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ for the respective bundles induced by these G-modules. All finite dimensional G-modules are submodules in tensor products of the fundamental representations V^α and V_α. Thus the bundles $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ play a special role and we term these *(local) twistor bundles* (c.f. [1,21]). In fact in line with the use of the word "tensor" we will also describe any explicit subbundle of a tensor product of these bundles as a twistor bundle and sections of such bundles as local twistors. In particular observe that there is a canonical completely skew local-twistor $(p+q)$-form $h_{\alpha\beta\cdots\gamma}$ on $\mathcal{E}^\alpha$ which is equivalent to the isomorphism (2). We write $h^{\alpha\beta\cdots\gamma}$ for the dual completely skew twistor satisfying $h^{\alpha\beta\cdots\gamma}h_{\alpha\beta\cdots\gamma} = (p+q)!$.

All finite dimensional P-modules enjoy filtrations which split completely as G_0-modules. V^α and V_α, give the simplest cases and, as P-modules, admit filtrations

$$V^\alpha = V^A + V^{A'}, \quad V_\alpha = V_{A'} + V_A.$$

(Our notational convention is that the 'right ends' in the formal sums are submodules while the 'left ends' are quotients.) These determine filtrations of the twistor bundles

$$\mathcal{E}^\alpha = \mathcal{E}^A + \mathcal{E}^{A'}, \quad \mathcal{E}_\alpha = \mathcal{E}_{A'} + \mathcal{E}_A.$$

We write $X^\alpha_{A'}$ for the canonical section of $\mathcal{E}^\alpha_{A'}$ which gives the injecting morphism $\mathcal{E}^{A'} \to \mathcal{E}^\alpha$ via

$$v^{A'} \mapsto X^\alpha_{A'} v^{A'}. \tag{11}$$

Similarly Y_α^A describes the injection of $\mathcal{E}_A$ into dual twistors,

$$\mathcal{E}_A \ni u_A \mapsto Y_\alpha^A u_A \in \mathcal{E}_\alpha. \tag{12}$$

It follows from standard representation theory that a choice of splitting of the exact sequence,

$$0 \to V^{A'} \to V^\alpha \to V^A \to 0$$

is equivalent to the choice of subgroup of P which is isomorphic to G_0. It follows immediately that a choice of splitting of the twistor bundle $\mathcal{E}^\alpha$ is equivalent to a reduction from $\mathcal{G}$ to $\mathcal{G}_0$. Such a splitting is a G_0-equivariant homomorphism $\xi : \mathcal{E}^\alpha \to \mathcal{E}^{A'}$. We can regard ξ here as a section of $\mathcal{E}_\alpha \otimes \mathcal{E}^{A'} = \mathcal{E}_\alpha^{A'}$ and then in our index notation the homomorphism is determined by $v^\alpha \mapsto \xi_\alpha^{A'} v^\alpha$, for any section v^α of $\mathcal{E}^\alpha$. The composition of ξ with the monomorphism $\mathcal{E}^{A'} \to \mathcal{E}^\alpha$ must be the identity so we have,

$$\xi_\beta^{A'} X_{B'}^\beta = \delta_{B'}^{A'}.$$

A splitting $\xi_\alpha^{A'}$ of $\mathcal{E}^\alpha$ determines a dual splitting λ_A^α of $\mathcal{E}_\alpha$, $\lambda_A^\alpha : \mathcal{E}_\alpha \to \mathcal{E}_A$. Given such splittings we have $\mathcal{E}_\alpha = \mathcal{E}^A \oplus \mathcal{E}^{A'}$ and $\mathcal{E}_\alpha = \mathcal{E}_{A'} \oplus \mathcal{E}_A$, so we may write sections of these bundles as 'matrices' such as

$$[u^\alpha]_\xi = \begin{pmatrix} u^A \\ u^{A'} \end{pmatrix} \in [\Gamma\mathcal{E}^\alpha]_\xi, \qquad [v_\alpha]_\xi = (v_A \;\; v_{A'}) \in [\Gamma\mathcal{E}_\alpha]_\xi.$$

We will always work with splittings determined by a choice of scale $\xi \in \mathcal{E}[1]$, as discussed earlier. (That we have used the same symbol as used for the kernel part of the symbol for the splitting is of course no accident. In fact the direct connection between the scale ξ and the corresponding section $\xi_\alpha^{A'}$ is given explicitly in Section 3.) If u^α and v_α, as displayed, are expressed by such a scale then the change of scale $\xi \mapsto \hat{\xi} = \Omega^{-1}\xi$ yields a transformation of these splittings. For example $[u^\alpha] \mapsto [u^\alpha]_{\hat{\xi}}$ where

$$[u^\alpha]_{\hat{\xi}} = \begin{pmatrix} \hat{u}^A \\ \hat{u}^{A'} \end{pmatrix} = \begin{pmatrix} u^A \\ u^{A'} - \Upsilon_B^{A'} u^B \end{pmatrix}.$$

With this understood we will henceforth drop the notation $[\cdot]_\xi$ and simply write, for example, $v_\alpha \mapsto \hat{v}_\alpha$ where

$$\hat{v}_\alpha = (\hat{v}_A \;\; \hat{v}_{A'}) = (v_A + \Upsilon_A^{B'} v_{B'} \;\; v_{A'}),$$

for the corresponding transformation of v^α. In particular, the objects $\xi_\alpha^{B'}$, λ_A^β are not invariant and

$$\hat{\xi}_\alpha^{B'} = \xi_\alpha^{B'} - Y_\alpha^A \Upsilon_A^{B'}, \qquad \hat{\lambda}_A^\beta = \lambda_A^\beta + X_{B'}^\beta \Upsilon_A^{B'}.$$

However, note that, in the splittings they determine, $\xi_\alpha^{B'}$ and λ_A^β are given

$$\xi_\alpha^{B'} = \begin{pmatrix} 0 & \delta_{A'}^{B'} \end{pmatrix}, \qquad \lambda_A^\beta = \begin{pmatrix} \delta_A^B \\ 0 \end{pmatrix}.$$

In any such splitting the invariant objects $X^{\alpha}_{B'}$ and Y^{A}_{β} are given by

$$X^{\alpha}_{B'} = \begin{pmatrix} 0 \\ \delta^{A'}_{B'} \end{pmatrix}, \quad Y^{A}_{\beta} = \begin{pmatrix} \delta^{A}_{B} & 0 \end{pmatrix}.$$

The first four identities of the following display are immediate, while the final two items are useful definitions:

$$\begin{array}{llll} Y^{A}_{\beta} X^{\beta}_{A'} = 0, & \xi^{A'}_{\beta} \lambda^{\beta}_{A} = 0, \\ Y^{A}_{\beta} \lambda^{\beta}_{B} = \delta^{A}_{B}, & \xi^{A'}_{\beta} X^{\beta}_{B'} = \delta^{A'}_{B'}, \\ Y^{A}_{\beta} \lambda^{\gamma}_{A} =: \lambda^{\gamma}_{\beta}, & \xi^{A'}_{\beta} X^{\gamma}_{A'} =: \xi^{\gamma}_{\beta}. \end{array} \tag{13}$$

We shall mostly deal with *weighted twistors*, i.e. tensor products of the form $\mathcal{E}^{\alpha\ldots\beta}_{\gamma\ldots\delta}[w] = \mathcal{E}^{\alpha\ldots\beta}_{\gamma\ldots\delta} \otimes \mathcal{E}[w]$. All the above algebraic machinery works for the weighted twistors. In fact we shall often omit the word 'weighted' even though, of course, these bundles do not come from G-modules for $w \neq 0$.

Finally, we observe that via this machinery any spinorial quantity may be identified with a (weighted) twistor. For example valence 1 spinors in $\mathcal{E}^{A'}[w_1]$ or $\mathcal{E}_A[w_2]$ may be dealt with via (11) or (12), respectively. This determines an identification for tensor powers by treating each factor in this way. This does all cases since, via (3),

$$\mathcal{E}^{A} \cong \mathcal{E}_{\underbrace{[B\cdots D]}_{q-1}}[1], \quad \mathcal{E}_{A'} \cong \mathcal{E}^{\overbrace{[B'\cdots C']}^{p-1}}[-1].$$

Now, any irreducible representation of G_0 is given as a tensor product of two irreducible components in tensor products of the fundamental spinors (viewed as representations of the special linear groups, adjusted by a weight). Applying the corresponding Young symmetrizers [10,20] to the tensor products of $\mathcal{E}_\alpha$ and $\mathcal{E}^\beta$, we obtain the explicit realization of each irreducible spinor bundle as the subbundle of the (weighted) twistor bundle which is isomorphic to the injecting part (in the composition series – see Appendix C) of the twistor bundle. Thus a section of a weighted irreducible spinor bundle $\mathcal{V}$ may be identified with a twistor object which is zero in all its composition factors except the first. So, in fact, this non-zero factor is also the projecting part of the twistor. We write $\tilde{\mathcal{V}}$ for this twistor (sub-)bundle satisfying $\mathcal{V} \cong \tilde{\mathcal{V}}$. Altogether, we have established the following result.

Lemma 2.2. *Any irreducible spinor object v can be identified with the twistor $\tilde{v}$ which has the spinor as its projecting part. This identification is provided in a canonical algebraic way.*

In this connection we may also talk about the algebraic construction providing the twistor bundle $\tilde{\mathcal{V}}$. In any concrete case the identifications may be described explicitly and in a rather obvious way using the projectors X, Y, λ, ξ.

3. Twistor calculus

Given a choice of scale ξ, a *twistor connection* ∇_a on $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ is given by the following formulae:

$$\nabla_A^{P'}\begin{pmatrix} v^B \\ v^{B'} \end{pmatrix} = \begin{pmatrix} \nabla_A^{P'} v^B + \delta_A^B v^{P'} \\ \nabla_A^{P'} v^{B'} - \mathsf{P}_{AB}^{P'B'} v^B \end{pmatrix} \tag{14}$$

and

$$\nabla_A^{P'}(u_B \; u_{B'}) = (\nabla_A^{P'} u_B + \mathsf{P}_{AB}^{P'B'} u_{B'} \;\; \nabla_A^{P'} u_{B'} - \delta_{B'}^{P'} u_A), \tag{15}$$

(cf. [1,9,19]). Notice that whereas on the left-hand side ∇ indicates the twistor connection, on the right-hand side the symbol ∇ indicates the usual spinor connection determined by the choice of scale. Although we have fixed a choice of scale to present explicit formulae for these connections, it is easily verified directly using the formulae (4) that the twistor connections are in fact independent of the choice of scale and so are invariant operators on the AG-structure.

An easy calculation reveals that

$$([\nabla_a, \nabla_b] - T_{ab}{}^d\nabla_d)\begin{pmatrix} v^C \\ v^{C'} \end{pmatrix} = \begin{pmatrix} U_{ab}{}_D^C v^D - T_{ab}{}_{D'}^C v^{D'} \\ -2\nabla_{[a}\mathsf{P}_{b]}{}_D^{C'} v^D + T_{ab}{}_{E'}^E \mathsf{P}_{ED}^{E'C'} v^D + U_{ab}{}_{D'}^{C'} v^{D'} \end{pmatrix}.$$

Thus the curvature of the twistor connection is given, in this scale, by

$$W_{ab}{}_\delta^\gamma = \begin{pmatrix} U_{ab}{}_D^C & -T_{ab}{}_{D'}^C \\ -2Q_{ab}{}_D^{C'} & U_{ab}{}_{D'}^{C'} \end{pmatrix}, \tag{16}$$

where

$$Q_{abc} := \nabla_{[a}\mathsf{P}_{b]c} - \tfrac{1}{2}T_{ab}{}^e\mathsf{P}_{ec}.$$

Note that since the twistor connection is invariant it follows that this *twistor curvature* $W_{ab}{}_\delta^\gamma$ is invariant. In fact, viewed as a $\mathfrak{g}$-valued 2-form on the Cartan bundle $\mathcal{G}$, this is just the curvature of the normal Cartan connection. In particular, we know that the structures are torsion-free (in the sense of the Cartan connection) if and only if the torsion part $T_{ab}{}^c$ vanishes and they are locally flat if and only if the whole $W_{ab}{}_\delta^\gamma$ vanishes.

The D-operators. Observe that if $f \in \mathcal{E}[w]$ then it follows easily from (5) that the spinor–twistor object

$$D_\beta^{A'} f := (\nabla_B^{A'} f \;\; w\delta_{B'}^{A'} f)$$

is invariant. We may regard this as an injecting part of the invariant twistor object $D_\beta^\alpha f := X_{A'}^\alpha D_\beta^{A'} f$. By regarding, in this formula for D_β^α, ∇ to be the coupled twistor–spinor connection it is easily verified that the operator D_β^α is well-defined and invariant on sections of the weighted twistor bundles $\mathcal{E}_{\alpha\cdots\gamma}^{\rho\cdots\mu}[w]$.

Definition 3.1. The invariant operators $D^\alpha_\beta : \mathcal{E}^{\rho\cdots\mu}_{\delta\cdots\gamma}[w] \to \mathcal{E}^{\alpha\rho\cdots\mu}_{\beta\delta\cdots\gamma}[w]$ are called the *twistor-D* operators.

For many calculations, where a choice of scale is made, it is useful to allow D^α_β to operate on spinors and their tensor products, although in this case the result is not independent of the scale. For example, if $v_C \in \mathcal{E}_C[w]$ then

$$D^{A'}_\beta v_C := (\nabla^{A'}_B v_C \;\; w\delta^{A'}_{B'} v_C).$$

Since the operator D^α_β and its concatenations will have an important role in the following discussions we develop notation for their target spaces. First let $\mathcal{F}^\rho$ be defined as follows:

$$\mathcal{F}^\rho := \ker(Y^A_\rho : \mathcal{E}^\rho \to \mathcal{E}^A).$$

Then we write

$$\mathcal{F}^{\rho\cdots\sigma}_{\alpha\cdots\beta} := \mathcal{F}^\rho \otimes \cdots \otimes \mathcal{F}^\sigma \otimes \mathcal{E}_\alpha \otimes \cdots \otimes \mathcal{E}_\beta,$$

and $\mathcal{F}^{\rho\cdots\sigma}_{\alpha\cdots\beta}[w] = \mathcal{F}^{\rho\cdots\sigma}_{\alpha\cdots\beta} \otimes \mathcal{E}[w]$. Finally let

$$\underbrace{\mathcal{S}^{\rho\,\cdots\,\sigma}_{\alpha\,\cdots\,\beta}}_{k}[w] := \left(\bigodot^k \mathcal{F}^\rho_\alpha\right) \otimes \mathcal{E}[w].$$

Note that sections of $\mathcal{F}^\alpha_\rho$ $(= \mathcal{S}^\alpha_\rho)$ are not generally trace-free, but that $\mathcal{F}^\alpha_\rho$ is in a complement to the trace-part of $\mathcal{E}^\alpha_\rho$.

Now if $f \in \mathcal{E}[w]$ then $D^\rho_\alpha f \in \mathcal{F}^\rho_\alpha[w]$. Similarly observe that if $v^\sigma \in \mathcal{F}^\sigma$ then

$$D^\rho_\alpha v^\sigma - \delta^\sigma_\alpha v^\rho$$

is in $\mathcal{F}^{\rho\sigma}_\alpha$. Thus

$$\mathsf{D}^{\rho\sigma}_{\alpha\beta} := \tfrac{1}{2}(D^\rho_\alpha D^\sigma_\beta + D^\sigma_\beta D^\rho_\alpha - \delta^\sigma_\alpha D^\rho_\beta - \delta^\rho_\beta D^\sigma_\alpha)$$

gives an invariant operator

$$\mathsf{D}^{\rho\sigma}_{\alpha\beta} : \mathcal{E}^{\mu\cdots\nu}_{\gamma\cdots\delta}[w] \to \mathcal{S}^{\rho\sigma}_{\alpha\beta} \otimes \mathcal{E}^{\mu\cdots\nu}_{\gamma\cdots\delta}[w].$$

Similarly we define $\mathsf{D}^{\rho\sigma\mu}_{\alpha\beta\gamma}$ by

$$\mathsf{D}^{\rho\sigma\mu}_{\alpha\beta\gamma} := \tfrac{1}{3}(D^\rho_\alpha \mathsf{D}^{\sigma\mu}_{\beta\gamma} + D^\sigma_\beta \mathsf{D}^{\rho\mu}_{\alpha\gamma} + D^\mu_\gamma \mathsf{D}^{\rho\sigma}_{\alpha\beta} - \delta^\rho_\alpha \mathsf{D}^{\sigma\mu}_{\beta\gamma} - \delta^\sigma_\beta \mathsf{D}^{\rho\mu}_{\alpha\gamma} - \delta^\mu_\gamma \mathsf{D}^{\rho\sigma}_{\alpha\beta})$$

and so on for $\mathsf{D}^{\alpha\cdots\delta}_{\rho\cdots\nu}$. Notice that the construction of these is designed in such a way that the resulting operators are annihilated if composed (contracted) with Y^B_ν on any index.

The splitting machinery. In terms of the algebraic projectors and embeddings introduced in Section 2, the twistor-D operator is given by

$$D^\rho_\alpha f = X^\rho_{R'} Y^A_\alpha \nabla^{R'}_A f + w\xi^\rho_\alpha f, \tag{17}$$

where f is any weighted twistor–spinor object. Using this and the expressions (14), (15) for the twistor connection, the following identities are easily established:

$$
\begin{array}{llll}
D^\rho_\alpha X^\beta_{C'} = X^\rho_{C'}\lambda^\beta_K Y^K_\alpha, & D^\rho_\alpha Y^C_\beta = -Y^C_\alpha X^\rho_{K'}\xi^{K'}_\beta, \\
D^\rho_\alpha \xi^{S'}_\beta = \mathsf{P}^{\rho S'}_{\alpha\beta}, & D^\rho_\alpha \lambda^\sigma_B = -\mathsf{P}^{\rho\sigma}_{\alpha B}, \\
X^\alpha_{B'} D^\gamma_\alpha f = w X^\gamma_{B'} f, & Y^B_\gamma D^\gamma_\alpha f = 0, \\
\xi^{B'}_\gamma D^\gamma_\alpha f = D^{B'}_\alpha f, & \lambda^\alpha_B D^{B'}_\alpha f = \nabla^{B'}_B f,
\end{array} \tag{18}
$$

where, again, f is any weighted twistor-spinor and we write $\mathsf{P}^{\rho\sigma}_{\alpha\beta} := \mathsf{P}^{R'S'}_{AB} X^\rho_{R'} X^\sigma_{S'} Y^A_\alpha Y^B_\beta$, $\mathsf{P}^{\rho S'}_{\alpha\beta} := \mathsf{P}^{R'S'}_{AB} X^\rho_{R'} Y^A_\alpha Y^B_\beta$, $\mathsf{P}^{\rho\sigma}_{\alpha B} := \mathsf{P}^{R'S'}_{AB} X^\rho_{R'} X^\sigma_{S'} Y^A_\alpha$, etc.

Notice also that the objects $\xi^{B'}_\alpha$ and λ^β_A describing the splitting of the twistors can be viewed as the projecting parts of $\xi^\beta_\alpha := \xi^{-1} D^\beta_\alpha \xi$ and $\delta^\beta_\alpha - \xi^\beta_\alpha$, respectively.

D-curvature. For $f \in \mathcal{E}[w]$ the projecting part of $D^\rho_\alpha f$ is $(1/p) X^\alpha_{P'} D^{P'}_\alpha f = wf$. Although this is 0th order in f, this part of $D^\rho_\alpha f$ behaves like a first order operator because of the weight factor, w. In particular $(1/p) X^\alpha_{P'} D^{P'}_\alpha$ satisfies a Leibniz rule and so therefore so does D^ρ_α. It follows immediately that, acting on $\mathcal{E}^\mu[w]$, $[D^\rho_\alpha, D^\sigma_\beta]$ decomposes into a 0th order curvature part and a 1st order torsion part. In fact it is an elementary exercise using the identities (8) and (18) to verify that

$$
[D^\rho_\alpha, D^\sigma_\beta] v^\mu = W^{\rho\sigma\mu}_{\alpha\beta\gamma} v^\gamma - W^{\rho\sigma\nu}_{\alpha\beta\gamma} D^\gamma_\nu v^\mu + \delta^\sigma_\alpha D^\rho_\beta v^\mu - \delta^\rho_\beta D^\sigma_\alpha v^\mu, \tag{19}
$$

where,

$$
W^{\rho\sigma\mu}_{\alpha\beta\gamma} = X^\rho_{A'} X^\sigma_{B'} Y^A_\alpha Y^B_\beta W^{A'B'}_{AB\gamma}{}^\mu. \tag{20}
$$

4. Invariant theory

Recall that each choice of scale determines the linear connection ∇^ξ on $\mathcal{G}_0$. We shall write $\Gamma_{(\xi)}$ for the coefficients of this connection ∇^ξ in some coordinate frame. The linear connections ∇^ξ are clearly expressed through the normal Cartan connection ω on $\mathcal{G}$ and vice versa (this is one of the important aspects of the Rho-tensor P_{ab}, cf. Appendix A). Thus we use the explicit definition of invariance given below. We use this approach for simplicity, but we should like to point out that there are more natural points of view fitting nicely into the general concepts as developed in [18]. In particular, some of our polynomiality assumptions follow then automatically.

Invariant operators. Let V and U be finite dimensional P-modules with V irreducible. A *(coupled) invariant operator* on $\mathcal{V}$ taking values in $\mathcal{U}$ is a well-defined differential operator $\mathcal{V} \to \mathcal{U}$ which may depend polynomially on the finite jets of the functions $\Gamma_{(\xi)}$ as well as polynomially on the finite jets of $\mathcal{V}$ and which is independent of the choice of local coordinate frame and scale ξ. By the very definition, such an invariant must be intrinsic to the AG-structure and so, when evaluated, depends only on the section of $\mathcal{V}$ and the normal Cartan connection ω of the structure. It is clearly sufficient to deal with the case that the invariant is homogeneous in $\mathcal{V}$ and we shall henceforth assume that coupled invariants are homogeneous in this way.

We say an invariant (and semi-invariants as described below) has order (l, m) if, in some scale ξ, it is:

(1) of order l as an operator on $v \in \Gamma(\mathcal{V})$ and,

(2) in any coordinates, as an operator on the functions $\Gamma_{(\xi)}$, it is of order $\geq m$ with equality in some set of coordinates.

We will also describe such an invariant as being of order k where $k := \max(l, m)$. In the special case that the invariant is homogeneous of degree 0 in the section v then it is an invariant of the structure. On the other hand, if the invariant is of degree 1 in v then it is a linear invariant operator on v.

If the invariant takes values in $\mathcal{U}$ where U is an irreducible P-module we will describe the invariant as an *irreducible invariant*. We may also restrict the definition of our operators to some subcategory of the structures in question. For example, we may require they are locally flat, or torsion free, etc.

We will show below that the twistor-D operator is a universal invariant differential operator in the sense that all coupled invariant operators arise in an appropriate sense from concatenations of this operator and its curvature.

Some examples. Note that the invariance of the exterior derivative on functions is implicit in the definition of the twistor-D operator. If $f \in \mathcal{E}$ then $D_\alpha^{P'} f = (\nabla_A^{P'} f \; 0)$ so the projecting part of $D_\alpha^\rho f$ is $\nabla_A^{P'} f$ and thus this is invariant. Similarly on

$$\begin{pmatrix} v^A \\ v^{A'} \end{pmatrix} = v^\alpha \in \mathcal{E}^\alpha,$$

$D_\alpha^{P'} v^\beta = (\nabla_A^{P'} v^\beta \;\; 0)$ and so the projecting part of $D_\alpha^\rho v^\beta$ is $\nabla_A^{P'} v^B + \delta_A^B v^{P'}$. The equation obtained by setting this to zero is the usual twistor equation [1].

For a second order example consider $\mathsf{D}_{\alpha\beta}^{\rho\sigma} f$ for $f \in \Gamma\mathcal{E}[w]$ (or $f \in \Gamma\mathcal{E}_{\alpha\cdots\gamma}^{\rho\cdots\mu}[w]$, with indices suppressed). It is easily established that

$$\mathsf{D}_{A\beta}^{P'\sigma} f = \begin{pmatrix} 0 & 0 \\ (\nabla_A^{P'} \nabla_B^{S'}) f + w \mathsf{S}_{AB}^{P'S'} f & w \delta_{B'}^{S'} \nabla_A^{P'} f - \delta_{B'}^{P'} \nabla_A^{S'} f \end{pmatrix}$$

and

$$\mathsf{D}_{A'\beta}^{P'\sigma} f = \begin{pmatrix} 0 & 0 \\ w \delta_{A'}^{P'} \nabla_B^{S'} f - \delta_{A'}^{S'} \nabla_B^{P'} f & w (w \delta_{A'}^{P'} \delta_{B'}^{S'} - \delta_{A'}^{S'} \delta_{B'}^{P'}) f \end{pmatrix}$$

and these two 'matrices' display all the non-vanishing parts of $\mathsf{D}_{\alpha\beta}^{\rho\sigma} f$. Here, and below, $(\nabla_a \nabla_b)$ means $\nabla_{(a} \nabla_{b)}$ and, recall, $S_{ab} := \mathsf{P}_{(ab)}$. If $w = 0$ the projecting part of this is $\nabla_B^{P'} f$. On the other hand if $w = 1$ then the projecting part of $\mathsf{D}_{\alpha\beta}^{(\rho\sigma)} f$ is

$$(\nabla_{(A}^{P'} \nabla_{B)}^{S'}) f + \mathsf{S}_{(AB)}^{P'S'} f$$

and so this is an invariant operator. Similarly if $w = -1$ then clearly the projecting part of $\mathsf{D}_{\alpha\beta}^{[\rho\sigma]} f$ is the invariant operator

$$(\nabla^{P'}_{[A}\nabla^{S'}_{B]})f - \mathsf{S}^{P'S'}_{[AB]}f.$$

In the case that $p = 2$ we may contract this with $\epsilon_{P'S'}$ to yield the invariant operator

$$\Box_{AB}f := \epsilon_{P'S'}((\nabla^{P'}_{A}\nabla^{S'}_{B}) - \mathsf{S}^{P'S'}_{AB})f. \tag{21}$$

If also $q = 2$ this is the usual conformally invariant Laplacian or Yamabe operator $(\Delta - \frac{1}{6}R)f$ where R is the Ricci scalar curvature.

On the other hand, if $w \neq -1, 0, 1$ then the projecting part of both $\mathsf{D}^{(\rho\sigma)}_{\alpha\beta}f$ and $\mathsf{D}^{[\rho\sigma]}_{\alpha\beta}f$ is a non-zero multiple of f. In fact it is an easy consequence of this observation and Theorem 4.4 that, on weighted functions of weight $w \neq -1, 0, 1$, there are no linear invariant operators of order ≤ 2 which are non-trivial on flat structures.

Note that in, for example, the $w = -1$ case above the operator $(\nabla^{P'}_{[A}\nabla^{S'}_{B]})f - \mathsf{S}^{P'S'}_{[AB]}f$ may be described explicitly by the formula

$$\xi^{P'}_{\rho}\xi^{S'}_{\sigma}\lambda^{\alpha}_{A}\lambda^{\beta}_{B}\mathsf{D}^{[\rho\sigma]}_{\alpha\beta}f.$$

It is useful to think of this as a composition of

$$\mathsf{D}^{[\rho\sigma]}_{\alpha\beta} : \mathcal{E}[-1] \to \mathcal{Q}^{\rho\sigma}_{\alpha\beta}$$

with

$$\xi^{P'}_{\rho}\xi^{S'}_{\sigma}\lambda^{\alpha}_{A}\lambda^{\beta}_{B} : \mathcal{Q}^{\rho\sigma}_{\alpha\beta} \to \mathcal{E}^{[P'S']}_{[AB]}[-1]. \tag{22}$$

Here $\mathcal{Q}^{\rho\sigma}_{\alpha\beta}$ is the minimal natural subbundle of $\mathcal{S}^{\rho\sigma}_{\alpha\beta}[-1]$ which contains the image of $\mathsf{D}^{[\rho\sigma]}_{\alpha\beta}$ on $\mathcal{E}[-1]$. This is induced by a P-submodule, say H, of the representation inducing $\mathcal{S}^{\rho\sigma}_{\alpha\beta}[-1]$ and the invariant map (22) arises from a P-homomorphism from H to U, where U is the P-module inducing $\mathcal{E}^{[P'S']}_{[AB]}[-1]$. It is clear that one can use such P-homomorphisms composed with the $\mathsf{D}^{(k)}$ operators to proliferate invariants. The content of Theorem 4.4 is that all invariants arise this way.

An easy proposition. We will observe here that via the twistor-D operator we obtain a special description of the jet bundle associated to twistor subbundles.

Let $\mathcal{V}$ be any subbundle of a weighted twistor bundle and let us write $\mathsf{D}^{(k)}$ for the linear differential operator

$$\mathsf{D}^{(k)} : \mathcal{V} \to (\mathcal{E} \oplus \mathcal{S}^{\rho}_{\alpha} \oplus \mathcal{S}^{\rho\sigma}_{\alpha\beta} \oplus \cdots \oplus \mathcal{S}^{\rho\cdots\mu}_{\underbrace{\alpha\cdots\gamma}_{k}}) \otimes \mathcal{V}$$

given by

$$f \mapsto f \oplus \mathsf{D}^{\rho}_{\alpha}f \oplus \mathsf{D}^{\rho\sigma}_{\alpha\beta}f \oplus \cdots \oplus \mathsf{D}^{\rho\cdots\mu}_{\alpha\cdots\gamma}f.$$

Recall that any kth order differential operator on a bundle factors through the associated bundle of k-jets. That is, any kth order linear invariant differential operator, taking values in a bundle $\mathcal{U}$, $\mathcal{V} \to \mathcal{U}$, is equivalent to a bundle morphism $J^k(\mathcal{V}) \to \mathcal{U}$. In particular, $\mathsf{D}^{(k)}$

factors through a linear mapping on the kth jet prolongation $J^k(\mathcal{V})$. The image of $\mathsf{D}^{(k)}$ fills a vector subbundle of $(\mathcal{E} \oplus \mathcal{S}^{\rho}_{\alpha} \oplus \mathcal{S}^{\rho\sigma}_{\alpha\beta} \oplus \cdots \oplus \mathcal{S}^{\rho\cdots\mu}_{\alpha\cdots\gamma}) \otimes \mathcal{V}$, which we denote by $\mathcal{J}^k(\mathcal{V})$.

Proposition 4.1. *Let $\mathcal{V}$ be any subbundle of a weighted twistor bundle of weight w. The operator $\mathsf{D}^{(k)}$ determines a bundle isomorphism,*

$$J^k(\mathcal{V}) \cong \mathcal{J}^k(\mathcal{V}).$$

Proof. In view of the definition of $\mathcal{J}^k(\mathcal{V})$, the operator $\mathsf{D}^{(k)}$ clearly determines a bundle epimorphism $J^k(\mathcal{V}) \to \mathcal{J}^k(\mathcal{V})$. That this is also injective follows by counting dimensions: Consider $f \in \mathcal{E}[w]$. Observe that the injecting part of

$$\underbrace{\mathsf{D}^{\rho\cdots\mu}_{\alpha\cdots\gamma}}_{k} f$$

is of the form

$$\underbrace{\nabla_{(a} \cdots \nabla_{d)}}_{k} f + \text{(lower order terms)}. \tag{23}$$

All other parts of $\mathsf{D}^{\rho\cdots\mu}_{\alpha\cdots\gamma} f$ are of order at most $k-1$ and so, by repeated use of (23), can be expressed polynomially in terms of

$$\underbrace{\mathsf{D}^{\rho\cdots\sigma}_{\alpha\cdots\beta}}_{l} f$$

for $l \le k-1$. Thus

$$\mathcal{J}^k(\mathcal{E}[w])/\mathcal{J}^{(k-1)}(\mathcal{E}[w]) \cong \left(\bigodot^k \mathcal{E}_a\right) \otimes \mathcal{E}[w]$$

but $(\odot^k \mathcal{E}_a) \otimes \mathcal{E}[w] \cong J^k(\mathcal{E}[w])/J^{k-1}(\mathcal{E}[w])$. In fact it is easily seen that, by an almost identical argument, we have the more general result,

$$\mathcal{J}^k(\mathcal{V})/\mathcal{J}^{(k-1)}(\mathcal{V}) \cong \left(\bigodot^k \mathcal{E}_a\right) \otimes \mathcal{V} \cong J^k(\mathcal{V})/J^{k-1}(\mathcal{V})$$

and so, by induction on k, the fibre dimension of $\mathcal{J}^k(\mathcal{V})$ is the same as the fibre dimension of $J^k(\mathcal{V})$. □

Although the proposition above is inspiring we need to consider slightly more general structures to obtain all invariants. These are defined before Theorem 4.4.

In the meantime we need to understand the general invariants of ∇_ξ, viewed as affine connections. In order to distinguish them from the AG-invariants, we will call these *semi-invariants* in the sequel.

Semi-invariants and their normal form. As above, let V and U be finite dimensional P-modules, with V-irreducible, and let $\mathcal{V}$ and $\mathcal{U}$ be the corresponding natural bundles. A *coupled semi-invariant operator* (which we will often abbreviate to *semi-invariant*) on $\mathcal{V}$ taking values in $\mathcal{U}$ is a universal formula which is polynomial in the coordinate derivatives of the functions $\Gamma_{(\xi)}$ and coordinate derivatives of the components of $\mathcal{V}$ (in some local frame) which is independent of the choice of local coordinates and frame (but may not be independent of the choice of scale ξ). Thus, for each choice of scale ξ, a semi-invariant is a differential operator $\mathcal{V} \to \mathcal{U}$ which may depend polynomially on the finite jets of the functions $\Gamma_{(\xi)}$ as well as polynomially on the finite jets of $\mathcal{V}$. Note that a coupled invariant operator is a coupled semi-invariant operator which, in addition, is independent of the choice of scale ξ. As for invariants, semi-invariants will be deemed *irreducible* if they take values in irreducible natural bundle.

It is easy to write down some examples of such semi-invariants. The curvature $R_{ab}{}^{c}_{d}$ and torsion $T_{ab}{}^{c}$ of ∇^{ξ} are polynomial in the finite jets of the functions $\Gamma_{(\xi)}$ and it is a classical result that these objects are tensorial and so are semi-invariants. Thus the irreducible parts of these tensors are irreducible semi-invariants. The objects $F_{ab}{}^{c}$ and $U_{ab}{}^{C}_{D}$ are examples. In fact due to the invariance of the covariant derivative it is easily verified that any contraction involving covariant derivatives of $v \in \Gamma(\mathcal{V})$ and covariant derivatives of the curvature $R_{ab}{}^{c}_{d}$ and the torsion T^{c}_{ab} is a semi-invariant. For example

$$v^{a}(\nabla_{a}v^{c})(\nabla_{c}U_{de}{}^{H}_{I})U_{fg}{}^{I}_{H}$$

is a semi-invariant. We will write $\mathrm{contr}(\nabla^{\xi}, T, R, v)$ to symbolically indicate such contractions. We will observe that, in fact, all semi-invariants arise this way and this leads to a standard way of expressing semi-invariants.

Let us fix a scale ξ. Note that it follows easily from Propositions B.1 and B.2 that a semi-invariant may be expressed as a polynomial in the components of the covariant derivatives of the torsion and curvature of $\nabla^{(\xi)}$ and the components of the covariant derivatives of the section $v \in \Gamma(\mathcal{V})$. At each point of the manifold a semi-invariant is a polynomial in the components of these tensors (that is the list of tensors which give, at that point, the various covariant derivatives of $T_{ab}{}^{c}$, $R_{ab}{}^{c}_{d}$ and v to the required order) which is covariant under the action of $\mathrm{SL}(p) \times \mathrm{SL}(q)$. Thus it follows from the complete reducibility of finite dimensional $(\mathrm{SL}(p) \times \mathrm{SL}(q))$-modules and Weyl's classical invariant theory [26] that *any* such semi-invariant can be expressed as a linear combination of basic semi-invariants of the form $\mathrm{contr}(\nabla^{\xi}, T, R, v)$ as claimed.

Consider then a semi-invariant expressed as a linear combination of contractions $\mathrm{contr}(\nabla^{\xi}, T, R, v)$. First observe that by substituting for $R_{ab}{}^{c}_{d}$ using formulae (6) and (7) we see that our typical semi-invariant may be re-expressed in terms of covariant derivatives of the objects v (with indices suppressed), $T_{ab}{}^{c}$, $U_{ab}{}^{C}_{D}$, $U_{ab}{}^{C'}_{D'}$ and P_{ab}. We might write $\mathrm{contr}(\nabla^{\xi}, T, U, U', \mathsf{P}, v)$ for the basic terms of this new expression. Finally we observe that the semi-invariant can be written as described in the following lemma which we regard as a *normal form* for semi-invariants.

Lemma 4.2. *Any semi-invariant of order k may be expressed as a linear combination of contractions involving the tensors* $S_{a\cdots d} \in \odot^{m}\mathcal{E}_{a}$ *for* $0 \le m \le k$, *and various covariant derivatives of the objects* $T_{ab}{}^{c}$, $U_{ab}{}^{C}_{D}$, $U_{ab}{}^{C'}_{D'}$, Q_{abc} *and* $v \in \Gamma(\mathcal{V})$.

Proof. Recall that

$$Q_{abc} = \nabla_{[a}\mathsf{P}_{b]c} - \tfrac{1}{2}T_{ab}{}^{e}\mathsf{P}_{ec}$$

(as in (16)). Note that it is easily verified, by considering possible Young symmetrizers and using (9), that the $(m-2)$nd covariant derivative of P_{ab},

$$\underbrace{\nabla_a \cdots \nabla_b}_{m-2} \mathsf{P}_{cd},$$

may, *up to lower order terms* which involve covariant derivatives of the curvature and torsion, be expressed as a linear combination of the tensors (cf. (9))

$$\underbrace{S_{a\cdots d}}_{m}, \quad \underbrace{\nabla_a \cdots \nabla_b}_{m-3} \nabla_{[c}\mathsf{P}_{d]e} \quad \text{and} \quad \underbrace{\nabla_a \cdots \nabla_b \nabla_e}_{m-1} T_{cd}{}^{e}.$$

Thus, by replacing $\nabla_{[c}\mathsf{P}_{d]e}$ with $Q_{cde} + \frac{1}{2}T_{cd}{}^{f}\mathsf{P}_{fe}$ it is clear that the tensors $\nabla_a \cdots \nabla_b \mathsf{P}_{cd}$ may, up to lower order terms which involve covariant derivatives of the curvature and torsion, be expressed as a linear combination of the tensors

$$\underbrace{S_{a\cdots d}}_{m}, \quad \underbrace{\nabla_a \cdots \nabla_b}_{m-3} Q_{cde} \quad \text{and} \quad \underbrace{\nabla_a \cdots \nabla_b \nabla_e}_{m-1} T_{cd}{}^{e}.$$

The lemma follows by first expressing the semi-invariant in the manner last described above and then repeatedly using this observation to replace all occurrences of covariant derivatives of P_{ab}, starting with the highest order. □

The main theorem. Note that the covariant derivatives of tensors and spinors can be expressed in terms of components of the covariant derivatives of P_{ab} and components of the twistor-D operator acting on appropriate twistors via the machinery of Section 3. For example, consider $\nabla_a v_B$ where $v_B \in \mathcal{E}_B[w]$. Let $v_\beta := Y^B_\beta v_B$ and we have

$$\nabla_a v_B = \xi^{A'}_{\rho} \lambda^{\alpha}_{A} D^{\rho}_{\alpha} \lambda^{\beta}_{B} v_\beta.$$

We now bring the λ^{β}_{B} to the left of the twistor-D operator using the appropriate identities from (18) and (13). We obtain

$$\nabla_a v_B = \xi^{A'}_{\rho} \lambda^{\alpha}_{A} \lambda^{\beta}_{B} (D^{\rho}_{\alpha} v_\beta).$$

Thus

$$\begin{aligned}
\nabla_a \nabla_b v_C &= \xi^{A'}_{\rho} \lambda^{\alpha}_{A} D^{\rho}_{\alpha} \xi^{B'}_{\sigma} \lambda^{\beta}_{B} D^{\sigma}_{\beta} \lambda^{\gamma}_{C} v_\gamma \\
&= \xi^{A'}_{\rho} \lambda^{\alpha}_{A} D^{\rho}_{\alpha} (\xi^{B'}_{\sigma} \lambda^{\beta}_{B} \lambda^{\gamma}_{C} (D^{\sigma}_{\beta} v_\gamma)) \\
&= \xi^{A'}_{\rho} \lambda^{\alpha}_{A} \xi^{B'}_{\sigma} \lambda^{\beta}_{B} \lambda^{\gamma}_{C} (D^{\rho}_{\alpha} D^{\sigma}_{\beta} v_\gamma) - \mathsf{P}^{A'C'}_{AB} X^{\beta}_{C'} \xi^{B'}_{\sigma} \lambda^{\gamma}_{C} (D^{\sigma}_{\beta} v_\gamma) \\
&\quad - \mathsf{P}^{A'C'}_{AC} X^{\gamma}_{C'} \xi^{B'}_{\sigma} \lambda^{\beta}_{B} (D^{\sigma}_{\beta} v_\gamma)
\end{aligned}$$

Continuing in this fashion it is easily seen that

$$\underbrace{\nabla_a \cdots \nabla_b}_{l} v_B$$

may be expressed in terms of components of

$$\underbrace{D^\rho_\alpha \cdots D^\sigma_\beta}_{l} v_\gamma$$

and lower order terms which polynomially involve the components of

$$\underbrace{D^\rho_\alpha \cdots D^\sigma_\beta}_{m} v_\gamma,$$

for $m \leq l-1$, and the components of covariant derivatives, to order $l-2$, of P_{ab}.

These observations lead us to the next lemma which is the key to the proof of the theorem in this section. As before let $\mathcal{V}$ be an irreducible natural bundle and recall that we may identify this with a twistor subbundle (Lemma 2.2). Since we are suppressing the indices on the section $v \in \mathcal{V}$ we will write $\tilde{v} \in \tilde{\mathcal{V}}$ for the corresponding section of the appropriate twistor bundle. The section v is recovered explicitly by contracting $\tilde{v}$ with the projectors $\xi^{A'}_\alpha$ and λ^α_A. (For example, in the example just above $v = v_B \in \mathcal{E}_B[w]$ and $\tilde{v} = v_\beta \in \mathcal{E}_\beta[w]$ with $v_B = \lambda^\beta_B v_\beta$.)

Lemma 4.3. *A coupled invariant differential operator I of order (l, m) may be expressed as a universal polynomial expression in the components of* $\mathsf{D}^{(l)}\tilde{v}$ *and* $\mathsf{D}^{(k')}W$ *where* $k' = \max(l-1, m)$.

In this lemma, and henceforth, $\mathsf{D}^{(m)}W$ means $\mathsf{D}^{(m)}$ applied to $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$. This is to be distinguished from $(D)^{(m)}W$ which we will use to mean simply an m-fold application of the twistor-D operator to $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$.

Proof. We may suppose that at first we have chosen a scale ξ and the invariant is expressed in normal form as in Lemma 4.2. We will first observe that, in rewriting this expression, covariant derivatives of $T_{ab}{}^c$, $U_{ab}{}^C_D$, $U_{ab}{}^{C'}_{D'}$, and Q_{abc} may be eliminated in favour of components of $\mathsf{D}^{(m)}W$ and lower order terms and similarly covariant derivatives of $v \in \Gamma(\mathcal{V})$ may be eliminated.

Note that each of the objects $T_{ab}{}^c$, $U_{ab}{}^C_D$, $U_{ab}{}^{C'}_{D'}$, and Q_{abc} may be obtained linearly from $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$ via the projectors $X^\alpha_{A'}$, Y^A_α, $\xi^{A'}_\alpha$, λ^α_A. For example

$$U_{ab}{}^C_D = \xi^{A'}_\rho \lambda^\alpha_A \xi^{B'}_\sigma \lambda^\beta_B Y^C_\gamma \lambda^\delta_D W^{\rho\sigma\gamma}_{\alpha\beta\delta}.$$

We re-express the invariant as follows. We make the substitutions for each of $T_{ab}{}^c$, $U_{ab}{}^C_D$, $U_{ab}{}^{C'}_{D'}$, Q_{abc} and v in terms of components of $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$ and $\tilde{v}$, and we replace each ∇_a with $\xi^{A'}_\rho \lambda^\alpha_A D^\rho_\alpha$. Next we further re-express by moving each $X^\alpha_{A'}$, Y^A_α, $\xi^{A'}_\alpha$ and λ^α_A to the left of any twistor-D operators. The new expression for the invariant involves components of concatenations of twistor-D operators acting on $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$ and $\tilde{v}$ and covariant derivatives of P_{ab} and various valence S-tensors. These covariant derivatives of P_{ab} all turn up via the identities (18). From this observation it is immediately clear that the order of any of these covariant derivatives of P_{ab} is strictly less than $k := \max(l, m)$. In fact, by elementary representation theory arguments, one can show that the order of any of these covariant derivatives of P_{ab} is $\leq l-2$ if $l > m$, and is $\leq m-2$ otherwise. Now we replace, in the last expression for the invariant, each maximal order $\nabla_a \cdots \nabla_c \mathsf{P}_{de}$ with its expression in terms of the tensors $S_{a\cdots e}$, $\nabla_a \cdots \nabla_b Q_{cde}$, $\nabla_a \cdots \nabla_c \nabla_f T_{de}{}^f$, their transposes and lower order terms. Next we replace each occurrence of $\nabla_a \cdots \nabla_b Q_{cde}$ and $\nabla_a \cdots \nabla_c \nabla_f T_{de}{}^f$ with their expressions in terms of components of $(D)^{k'} W$ and lower order covariant derivatives of P_{ab}. Continuing in this fashion it is clear that finally we are left with an expression involving only S-tensors and the components of concatenations of the twistor-D operator on $W^{\rho\sigma\gamma}_{\alpha\beta\delta}$ and $\tilde{v}$. It is easily seen using (19) that this may be re-expressed in terms of components of $\mathsf{D}^{(l)}\tilde{v}$, $\mathsf{D}^{(m)}W$ and the components of the S-tensors.

Now let us write the invariant I as a sum of two parts

$$I = A + B,$$

where the part A consists of all terms which involve no components of the S-tensors while B is the remaining part which consists of all terms which do involve the S-tensors. Let us choose a point q and consider changing the scale of ξ by a factor Ω so that $\Upsilon_a(q) = 0$. Under such a transformation it is clear that, at q, the A part of I is invariant as the transformation of the components of an invariant twistor depends only on the first derivative of Ω. On the other hand, I is invariant under any transformation. Thus it follows that, under transformations such that $\Upsilon_a(q) = 0$, B must also be invariant. But on the other hand, B vanishes in a normal scale ξ_q (see (B.3) in Section B) since in this scale all the S-tensors vanish at q. As observed in Remark B.3, such a scale can be achieved by a transformation with $\Upsilon_a(q) = 0$. Thus B must vanish at q. Since we may perform this calculation at any point it follows that B vanishes everywhere so $I = A$ and the lemma is proved. □

Before we can discuss the main theorem we will need some special notation. Recall that $\mathcal{J}^l(\mathcal{V})$ was defined to be the subbundle of the natural bundle

$$\mathcal{V}^{(l)} := (\mathcal{E} \oplus \mathcal{S}^{\rho}_{\alpha} \oplus \mathcal{S}^{\rho\sigma}_{\alpha\beta} \oplus \cdots \oplus \underbrace{\mathcal{S}^{\rho\cdots\mu}_{\alpha\cdots\gamma}}_{l}) \otimes \tilde{\mathcal{V}}$$

determined by the image of the invariant operator $\mathsf{D}^{(l)}$ on $\tilde{\mathcal{V}}$. Note that in general $\mathcal{J}^l(\mathcal{V})$ will not itself be a natural bundle as the algebraic properties of its fibres vary over M. Suppose that μ and $V^{(l)}$ denote, respectively, the P-representation and representation space inducing the natural bundle $\mathcal{V}^{(l)}$ displayed. If $\tilde{v}$ is a section of $\tilde{\mathcal{V}}$ then $\mathsf{D}^{(l)}(\tilde{v})$ is a section of $\mathcal{J}^l(\mathcal{V})$. That is $\mathsf{D}^{(l)}(\tilde{v})$ is a function

$$\mathsf{D}^{(l)}(\tilde{v}) : \mathcal{G} \to V^{(l)},$$

which is homogeneous, $\mathsf{D}^{(l)}(v)(x \cdot p) = \mu(p^{-1})\mathsf{D}^{(l)}(v)(x)$, $x \in \mathcal{G}$ and $p \in P$. Note that in general this function is not surjective. Note also that the image of $\mathsf{D}^{(l)}$ depends on the underlying structure of M, that is on the normal Cartan bundle equipped by the normal Cartan connection.

Our next step is to construct a sort of smallest natural bundle which could accommodate the values of $\mathsf{D}^{(l)}$. Let us fix a point $q \in M$ and a coordinate neighbourhood $Q = \mathbb{R}^{pq}$ centred at q (in fact we may forget about our manifold M and we work just over $\mathbb{R}^{pq}$ for the while). Let us consider all possible normal Cartan connections on the trivial Cartan bundle $\mathcal{G} = Q \times P$ and for each such normal Cartan connection ω, let us write

$$\mathcal{J}^l_o(\mathcal{V}, \omega)$$

to denote the span of the image of $\mathsf{D}^{(l)}(\tilde{v})$, on the fibre of $\mathcal{G}$ over q, as we vary over all possible argument sections v. Note that $\mathcal{J}^l_o(\mathcal{V}, \omega)$ is a well-defined P-submodule of $V^{(l)}$. Now let

$$\mathcal{J}^l_o(\mathcal{V}) := \langle \cup_\omega \mathcal{J}^l_o(\mathcal{V}, \omega)\rangle,$$

the span of the union which is taken over all possible normal Cartan connections on $\mathcal{G} \to Q$. Then $\mathcal{J}^l_o(\mathcal{V})$ is also a well-defined P-submodule of $V^{(l)}$ and the corresponding natural subbundle in $\mathcal{V}^{(l)}$ is the smallest one containing all possible subbundles $\mathcal{J}^l(\mathcal{V})$.

Next we observe that we can consider a similar 'generic natural bundle' for the curvature. Write $W^{(m)}$ for the P-module inducing

$$(\mathcal{E} \oplus \mathcal{S}^\rho_\alpha \oplus \mathcal{S}^{\rho\sigma}_{\alpha\beta} \oplus \cdots \oplus \underbrace{\mathcal{S}^{\rho\cdots\mu}_{\alpha\cdots\gamma}}_{m}) \otimes \mathcal{E}^{\upsilon\phi\zeta}_{\delta\epsilon\vartheta}.$$

Then, for each normal Cartan connection, $\mathsf{D}^{(m)}W$ takes values in $W^{(m)}$ and the span of the image of $\mathsf{D}^{(m)}W$ on the fibre of $\mathcal{G}$ over $q \in M$ is a P-submodule of $W^{(m)}$ that we will denote $\mathcal{J}^m_o(\mathcal{W}, \omega)$. In analogy with the above we let

$$\mathcal{J}^m_o(\mathcal{W}) := \langle \cup_\omega \mathcal{J}^m_o(\mathcal{W}, \omega)\rangle,$$

here the union is taken over all possible normal Cartan connections defined locally at the fixed point $q \in M$. This is clearly a P-submodule of $W^{(m)}$. Now write

$$\mathcal{J}^{m,t}_o(\mathcal{W}) := \bigoplus_{i=0}^{t} \odot^i \mathcal{J}^m_o(\mathcal{W}).$$

What we really need is a 'generic fibre' for a mixed case. Write

$$\mathcal{J}^{l,s,m,t}_o(\mathcal{V}, \mathcal{W})$$

for the P-submodule of $(\odot^s V^{(l)}) \otimes (\oplus^t_{i=0} \odot^i W^{(m)})$ determined by the image of $(\odot^s(\mathsf{D}^{(l)}v)) \otimes (\oplus^t_{i=0} \odot^i \mathsf{D}^{(m)}W)$, at $q \in M$, as we vary the normal Cartan connection and for each such connection vary $\tilde{v}$ over all possible sections of $\tilde{\mathcal{V}}$. Note that $\mathcal{J}^{l,s,m,t}_o(\mathcal{V}, \mathcal{W})$ is clearly a P-submodule of $(\odot^s \mathcal{J}^l_o(\mathcal{V})) \otimes \mathcal{J}^{m,t}_o(\mathcal{W})$. Note also that the previous 'generic fibres' are special cases of this, $\mathcal{J}^{l,s,m,0}_o(\mathcal{V}, \mathcal{W}) = \odot^s \mathcal{J}^l_o(\mathcal{V})$ and $\mathcal{J}^{l,0,m,t}_o(\mathcal{V}, \mathcal{W}) = \mathcal{J}^{m,t}_o(\mathcal{W})$.

The following is the main theorem of this section.

Theorem 4.4. *A coupled invariant operator I which is homogeneous of degree s on an irreducible bundle $\mathcal{V}$, and taking values in the natural bundle $\mathcal{U}$, is equivalent to a P-homomorphism*

$$I_o : \mathcal{J}_o^{l,s,m,t}(\mathcal{V}, \mathcal{W}) \to U.$$

That is there is a 1–1 correspondence between such invariant operators I and homomorphisms I_o as indicated.

Proof. ($\Rightarrow$) Let $I(v, \omega)$ indicate the invariant I evaluated on a section v of $\mathcal{V}$ and some particular normal Cartan connection. Then $I(v, \omega) : \mathcal{G} \to U$ with the homogeneity property $I(v, \omega)(x \cdot p) = \sigma(p^{-1})I(v, \omega)(x)$ where σ denotes the inducing representation of P on U.

Lemma 4.3, combined with standard polarization techniques, implies that there is a linear mapping $\tilde{I}$, defined on the whole P-module $\odot^s V^{(l)} \otimes (\oplus_{i=0}^t \odot^i W^{(m)})$ (notice that our operator is homogeneous in the arguments from $\mathcal{V}$) such that

$$I(v, \omega)(x) = \tilde{I}\left(\left(\bigodot^s (\mathsf{D}^{(l)}\tilde{v})\right) \otimes \left(\bigoplus_{i=0}^t \bigodot^i \mathsf{D}^{(m)} W(\omega)\right)(x)\right) \tag{24}$$

for all $x \in \mathcal{G}$. The mapping $\Phi(v, \omega) : \mathcal{G} \to \mathcal{J}_o^{l,s,m,t}(\mathcal{V})$,

$$x \mapsto \left(\bigodot^s (\mathsf{D}^{(l)}\tilde{v})\right) \otimes \left(\bigoplus_{i=0}^t \bigodot^i \mathsf{D}^{(m)} W(\omega)\right)(x)$$

is P-equivariant too, and a general element in $\mathcal{J}_o^{l,s,m,t}(\mathcal{V})$ is a finite linear combination $\sum_j c_j \Phi(v_j, \omega_j)$. Since $\tilde{I}$ is linear, the equivariance of the compositions $I(v_j, \omega_j) = \tilde{I} \circ \Phi(v_j, \omega_j)$ implies that the restriction I_o of $\tilde{I}$ to $\mathcal{J}_o^{l,s,m,t}(\mathcal{V})$ is P-equivariant, as required. This shows that all invariants I arise from a P-homomorphism as in the theorem.

($\Leftarrow$) The composition of I_o with $(v, \omega) \mapsto (\odot^s(\mathsf{D}^{(l)}\tilde{v})) \otimes (\oplus_{i=0}^t \odot^i \mathsf{D}^{(m)} W)$, for each $v \in \Gamma(\mathcal{V})$ and normal Cartan connection ω, is clearly a coupled invariant operator. This shows that all invariants I arise from a P-homomorphism as in the theorem. We complete the proof by showing that if $I_o \neq 0$ then $I \neq 0$. If $I_o \neq 0$ then there exists an element of $\mathcal{J}_o^{l,s,m,t}(\mathcal{V}, \mathcal{W})$ such that I_o does not kill this element. As mentioned above a general element of $\mathcal{J}_o^{l,s,m,t}(\mathcal{V}, \mathcal{W})$ may be expressed by a finite linear combination $\sum_j c_j \Phi(v_j, \omega_j)$. Let this finite linear combination represent, in particular, the element not killed by I_o. It follows that one of the $\Phi(v_j, \omega_j)$ is not killed by I_o. That is there exists a section $v = v_j$ and a Cartan connection $\omega = \omega_j$ such that $I_o(\Phi(v, \omega)) \neq 0$. But this means the invariant differential operator which is I_o composed with Φ is non-trivial. Thus we have shown that any non-trivial P-homomorphism yields a non-trivial invariant operator as required. □

5. New invariant operators

As we have discussed, the case $p = q = 2$ corresponds to the usual four-dimensional conformal structures. Here we restrict attention to torsion-free AG-structures with $p = 2$, $q > 2$. The main result is Theorem 5.1 which, for these geometries, gives curved analogues for all the non-standard operators between differential forms. We will deal with $q > 2$ odd as well as even, but we would like to point out that the cases of q even are of particular interest as these include all the quaternionic geometries. That is, when q is even our formulae and results below describe invariant operators on the quaternionic geometries. This is in the spirit of our simultaneous treatment of AG-structures, so of course the formulae also give invariant operators for the other geometries (i.e. those corresponding to the 'real split form' $\mathrm{SL}(p+q, \mathbb{R})$ and we obtain similar operators when q is odd.

At this point it is worthwhile to review the examples exposed above and in particular the operator $\Box_{AB} f$ as displayed in (21). Although this second order invariant operator does not operate between forms it is closely related to the fourth order operators we construct below.

To describe the operators it is useful to have some efficient and concise notation for the bundles concerned. For this we will use Young diagrams [10,20]. We will use these to indicate projections onto irreducible representations of $\mathrm{SL}(m)$. In our case we will in particular use these for representations of $\mathrm{SL}(p+q)$, representations of $\mathrm{SL}(p) \times \mathrm{SL}(q)$, which are trivial with respect to the $\mathrm{SL}(p)$ factor, and the bundles these induce. (Here, as usual, $\mathrm{SL}(r)$ can mean either $\mathrm{SL}(r, \mathbb{R})$ or $\mathrm{SL}(r, \mathbb{C})$ depending on which structures we are considering. The comments here apply equally to both cases.) For example we could write $\square\!\square\,(\otimes^2 \mathcal{E}_A)$ or $\square\!\square\,\mathcal{E}_{AB}$ to mean $\mathcal{E}_{(AB)}$. In fact we will shorten this notation further and simply write $\square\!\square\,\mathcal{E}_A$ for this, that is $\square\!\square\,\mathcal{E}_A = \mathcal{E}_{(AB)}$. In this notation the total number of boxes in the given Young diagram indicates the required tensor power of the bundle. For diagrams of height and width greater than 1 we adopt the convention that we symmetrized over sets of indices corresponding to rows of the diagram first and then with the result skew over sets of indices corresponding to the columns of the diagram. For instance suppose we start with some general valence 3 spinor $A_{EFG} \in \mathcal{E}_{EFG}[w]$. If we first symmetrized over the last two indices to form $B_{EFG} := A_{E(FG)}$ and then on this result skew on the first two indices to obtain $C_{EFG} := B_{[EF]G}$, then

$$C_{EFG} \in (\begin{smallmatrix}\square&\!\square\\\square&\end{smallmatrix}\,\mathcal{E}_F)[w].$$

Although, for the sake of being concrete, we will suppose that this is the convention adopted, nothing we do actually depends on this choice of convention.

We will use this notation immediately in the construction of a special invariant operator. Recall that if $\mathcal{E}^*[w]$ is any weighted twistor bundle then we have the invariant operator $\mathsf{D}_{\alpha\beta\gamma\delta}^{\rho\sigma\mu\nu} : \mathcal{E}^*[w] \to \mathcal{F}_{\alpha\beta\gamma\delta}^{\rho\sigma\mu\nu} \otimes \mathcal{E}^*[w]$. Equivalently we may view this as an operator

$$D_{\alpha\beta\gamma\delta}^{R'S'U'V'} : \mathcal{E}^*[w] \to \mathcal{E}_{\alpha\beta\gamma\delta}^{R'S'U'V'} \otimes \mathcal{E}^*[w].$$

Thus there is an invariant operator

$$D_{\alpha\beta\gamma\delta} := \epsilon_{R'S'}\epsilon_{U'V'} D_{\alpha\beta\gamma\delta}^{R'S'U'V'} : \mathcal{E}^*[w] \to \mathcal{E}_{\alpha\beta\gamma\delta} \otimes \mathcal{E}^*[w-2].$$

Note that $D_{\alpha\beta\gamma\delta}$ inherits some symmetry from $\mathsf{D}^{\rho\sigma\mu\nu}_{\alpha\beta\gamma\delta}$, in particular observe that $D_{\alpha\beta\gamma\delta} = D_{[\alpha\beta][\gamma\delta]}$. For any $0 \le k \le q-2$ and weight $w \in \mathbb{R}$, let us write $\Box_{\alpha\beta\gamma\delta}$ for the non-trivial composition of

$$D_{\alpha\beta\gamma\delta} : \left(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_\sigma \right)[w] \to \left(\mathcal{E}_{\alpha\beta\gamma\delta} \otimes \left(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_\sigma \right)[w-2] \right)$$

with a Young projection

$$\left(\mathcal{E}_{\alpha\beta\gamma\delta} \otimes \left(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_\sigma \right)[w-2] \to \left(k+2\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_\alpha \right)[w-2] \right).$$

This is clearly invariant for all w. Note also that it is an elementary exercise to verify that there is such a composition which is non-trivial and that it is unique up to a natural isomorphism of the image bundle. (For example, in the $k = 0$ case the main point is to observe that $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array}$ turns up precisely once in the product $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$.)

Before we state the theorem let us introduce one further item of notation. Let us write $\mathcal{H}_\alpha$ for the subbundle of $\mathcal{E}_\alpha$ which is naturally isomorphic to $\mathcal{E}_A$ (c.f. $\mathcal{F}^\alpha$ of Section 3). Here is the main result of this section.

Theorem 5.1. *Let M be a torsion-free AG-structure, $p = 2$, $q > 2$. For each integer k such that $0 \le k \le q-2$ there is a fourth order invariant operator,*

$$\Box_{ABCD} : \left(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_E \right)[-k] \to \left(k+2\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{E}_E \right)[-k-2],$$

which is non-trivial on flat structures.

For each k the operator is given by

$$\Box_{\alpha\beta\gamma\delta} : \left(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{H}_\varepsilon \right)[-k] \to \left(k+2\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\,\mathcal{H}_\varepsilon \right)[-k-2].$$

Before entering the proof of this theorem, we shall discuss the corresponding operators on the locally flat AG-structures since their existence is a key to our proof below.

Remark 5.2. *The structure of linear invariant operators on the locally flat geometries is well understood in the literature. In particular, it follows from the theory of generalized Verma modules that, for each k as in the theorem, there is exactly one non-trivial operator, up to scalar multiples, between the bundles in question, see e.g.* [4].

It is straightforward to deduce formulae for these operators: First observe that there are preferred scales in the flat geometries, namely those with $\mathsf{P}_{ab} = 0$. The covariant derivatives commute for such scales ξ, and we will express our formulae in such a scale. Now for all $0 \le l \le q-2$ the bundle

$$\left(\updownarrow^{l}\ \boxplus\,\mathcal{E}_A\right)[-l]$$

is an irreducible component of the $2l$-forms on M (appearing with mutliplicity 1) and the operators between the bundles in the theorem are precisely the non-standard operators in the BGG-resolution of the functions, cf. the diagram in the end of Appendix A. Thus it is clear that the operators concerned are fourth order. At the same time, since no primed indices appear explicitly in our target modules, the operators must be given by $\nabla_{ABCD} := \epsilon_{A'B'}\epsilon_{C'D'}\nabla_A^{A'}\nabla_B^{B'}\nabla_C^{C'}\nabla_D^{D'}$, followed by an appropriate G_0-module homomorphism onto the target. (In the preferred scales all curvature vanishes so there is no possibilty of adding lower order terms.) Since the covariant derivatives commute there is only one non-trivial way to apply the Young projection $\boxplus$ to the image of the operator ∇_{ABCD}. Let us write $\tilde{\Box}_{ABCD}$ for the composition of such a Young projection with the operator ∇_{ABCD}, followed by the (again unique up to multiple) projection onto the desired target. It is clear then, that in such a preferred scale for the flat case, the operators of the theorem are given explicitly by the operator $\tilde{\Box}_{ABCD}$.

Proof of Theorem 5.1. Since $\Box_{\alpha\beta\gamma\delta}$ is invariant we have only to demonstrate the claim of the second part of the theorem, namely that for each k as in the theorem and upon restriction to the subbundle

$$\left(\updownarrow^{k}\ \boxplus\,\mathcal{H}_\sigma\right)[-k] \cong \left(\updownarrow^{k}\ \boxplus\,\mathcal{E}_S\right)[-k]$$

of

$$\left(\updownarrow^{k}\ \boxplus\,\mathcal{E}_\sigma\right)[-k]$$

the invariant operator $\Box_{\alpha\beta\gamma\delta}$ takes values in the subbundle

$$\left(\updownarrow^{k+2}\ \boxplus\,\mathcal{H}_\alpha\right)[-k-2] \cong \left(\updownarrow^{k+2}\ \boxplus\,\mathcal{E}_A\right)[-k-2]$$

of $(\updownarrow^{k+2}\ \boxplus\,\mathcal{E}_\alpha)[-k-2]$. On the way we shall also prove that, upon restriction to the flat structures, the resulting operator coincides with the known invariant operator on homogeneous structures and so it is non-trivial and fourth order. The combination of these results establishes the theorem.

First we will do the whole task for flat AG-structures. For this case let us restrict to a scale ξ such that $\mathsf{P}_{ab} = 0$ on M. It follows immediately from the definition of $\Box_{\alpha\beta\gamma\delta}$ in terms of D_α^ρ and the definition of the latter in terms of $\nabla_A^{R'}$ (see (17)) that the injecting part of

$$\Box_{\alpha\beta\gamma\delta}:\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{H}_\rho\right)[w]\to\left(k+2\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_\alpha\right)[w-2] \tag{25}$$

is a fourth order operator which, up to a constant non-zero scale, is a composition of

$$\nabla_{ABCD}:\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_R\right)[w]\to(\mathcal{E}_{ABCD})\otimes\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_R\right)[w-2]$$

with a Young projection

$$(\mathcal{E}_{ABCD})\otimes\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_R\right)[w-2]\to\left(k+2\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_A\right)[w-2].$$

Since $k+2\leq q$ the symmetries enjoyed by this are precisely the symmetries of (25) if one formally identifies the twistor indices of (25) with the upper case Roman indices in the obvious way. Now according to the comments in the remark above (and given our choice of scale ξ), up to scale, all such Young projections yield the same *non-trivial* fourth order operators. In particular, the injecting part of the image

$$\Box_{\alpha\beta\gamma\delta}\left(\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{H}_\rho\right)[-k]\right) \tag{26}$$

is a non-zero scalar multiple of the invariant operator $\Box_{ABCD}$ in flat AG-structures. That is for

$$f_{\rho\cdots\sigma}\in\Gamma\left(\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{H}_\rho\right)[-k]\right),$$

we have that

$$\lambda_A^\alpha\cdots\lambda_D^\gamma\lambda_R^\rho\cdots\lambda_S^\sigma\Box_{\alpha\cdots\gamma}f_{\rho\cdots\sigma}$$

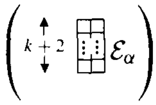
is independent of the choice of scale ξ (which recall determines λ_A^β) from within the preferred class of scales that have $\mathsf{P}_{ab}=0$. It follows that

$$f_{\rho\cdots\sigma}\mapsto(\Box_{\alpha\cdots\gamma}f_{\rho\cdots\sigma}-(Y_\alpha^A\cdots Y_\gamma^D Y_\rho^R\cdots Y_\sigma^S)(\lambda_A^{\alpha'}\cdots\lambda_D^{\gamma'}\lambda_R^{\rho'}\cdots\lambda_S^{\sigma'})\Box_{\alpha'\cdots\gamma'}f_{\rho'\cdots\sigma'})$$

gives an invariant operator

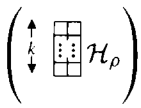
$$\left(k\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{H}_\rho\right)[-k]\to\left(k+2\updownarrow\ \begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \hline\end{array}\,\mathcal{E}_\alpha\right)[-k-2].$$

It is immediately clear that this invariant operator is annihilated upon contraction with $\lambda_A^\alpha\cdots\lambda_S^\sigma$, where these λ_A^α's are determined by any scale ξ' from the preferred class. (That is we do not need $\xi'=\xi$ as the operator is independent of the choice of scale.) Thus the operator vanishes when composed with any such projection onto the first composition factor of

 $[-k-2]$.

It follows immediately from Theorem C.1 of Appendix C that the operator itself must vanish and so the theorem is established for flat structures.

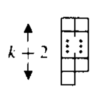
It is clear from this result for the flat case, that on general (or curved) structures the principal part of the operator $\Box_{\alpha\beta\gamma\delta}$ on

 $[-k]$

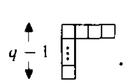
is non-vanishing and has image in the required bundle. Now let us fix a point $q \in M$ and a normal scale ξ_q (see Appendix B) and consider, at q, the composition of this operator with a projection to an irreducible part of the second composition factor. Notice that our choice excludes all occurrences of symmetrized derivatives of the Rho-tensors (that is the S-tensors), since these vanish under our choices. A typical result is given by

$$X^{\alpha}_{A'}\lambda^{\beta}_{B}\cdots\lambda^{\sigma}_{S}\Box_{\alpha\beta\gamma\delta}f_{\rho\cdots\sigma}. \tag{27}$$

Such a part of the operator must vanish in the flat case and so can only involve the curvature and its covariant derivatives contracted into covariant derivatives of the section $f_{R\cdots S}$. The unprimed indices of this carry a Young symmetry of the type



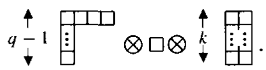
Now, recall we are considering only torsion-free AG-structures. Thus, as discussed in Appendix A, the only non-zero irreducible component in the $\mathfrak{g}_0$-part of the curvature W of the normal Cartan connection is the completely trace-free spinor $W_{ABC}^{A'B'D} = \tilde{U}_{ABC}^{A'B'D} = \tilde{U}_{(ABC)}^{[A'B']D}$ as the other parts vanish. This is equivalent to $W_{AB\cdots F} := W_{ABC}^{A'B'D}\epsilon_{A'B'}\epsilon_{DE\cdots F}$ which we will call the Weyl spinor. Observe that this has a Young symmetry

.

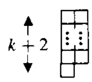
The $\mathfrak{g}_1$-part of the curvature W may be expressed polynomially and purely in terms of the first derivatives of the latter Weyl spinor (see Appendix A). Now, from order considerations and classical invariant theory it is clear that the typical term (27) must be a linear combination of contractions of the terms

$$(\nabla^{A'}_{A}W_{BC\cdots E})f_{G\cdots H} \quad \text{and} \quad W_{BC\cdots E}\nabla^{A'}_{A}f_{G\cdots H}. \tag{28}$$

Considering only the unprimed indices, these terms take values in representations of $\mathrm{SL}(q)$ described by the tensor product of Young tableaux

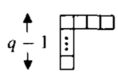


However, we claim that the diagram



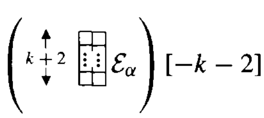
(29)

cannot turn up in this tensor product. To see this note that the only way that one could arrive at the diagram (29) by adding boxes to the diagram



is by first producing two full columns, then a further $2k+1$ boxes in appropriate positions. Finally further full columns could be added. But, since $q > 2$, for any non-negative integer l, $q+2+2k+1 \neq 2q+2k+3+lq$ so the outcome is impossible.

Thus the part (27) of the operator must vanish and, by the same argument, all irreducible parts of second composition factor (i.e. one away from the injecting part) must vanish. Thus, by the result (of Appendix C) that in any composition series (C.1) $V_t = 0 \Longrightarrow V_{t+1} = 0$ combined with Corollary C.3, it follows that the operator (26) must take values in the first composition factor in the bundle



and the theorem is proved. □

6. Further observations and remarks

As mentioned above local twistors for four-dimensional conformal spin structures have been described and investigated by Penrose and others [9,19,21]. Analogous local twistor bundles for complex AG-structures were defined by Bailey and Eastwood in [1]. The key to our progress here is the twistor-D operator of Definition 3.1. This enables a 'differentiation' which acts between local twistor bundles. Although this operator is new, it is very closely related to an operator D_{AP} between the so-called tractor bundles of conformal geometry as described in [14,15]. Much of the calculus surrounding the tractor bundles goes back to Tracy Thomas whose ideas were recovered and extended in [2]. We will not elaborate in detail on these connections in the current work. However, we briefly indicate here how the twistor-D operator may be used to define a *tractor-D operator* for AG-structures which agrees with the usual tractor-D operator, as described in [2], on four-dimensional conformal spin geometries.

The Tractor Calculus. Let us recall the natural bundles $\mathcal{E}^\alpha \supset \mathcal{F}^\alpha \simeq \mathcal{E}^{A'}$. Thus there is the tautological object $X^{\rho\cdots\sigma}$ providing the identification of the top degree exterior product of $\mathcal{F}^\alpha$ with a line bundle:

$$\mathcal{F}^{\overbrace{[\rho\cdots\sigma]}^{p}} = X^{\rho\cdots\sigma}\mathcal{E}[-1]. \tag{30}$$

(In fact $X^{\rho\cdots\sigma} = X^{\rho}_{R'}\cdots X^{\sigma}_{S'}\epsilon^{R'\cdots S'}$.) We define a *tractor-D* operator, $D_{\alpha\cdots\beta}$, as follows:

$$X^{\rho\cdots\sigma}D_{\alpha\cdots\beta}f := \mathsf{D}^{\overbrace{[\rho\cdots\sigma]}^{p}}_{\alpha\cdots\beta}f,$$

for f (with indices suppressed) in $\mathcal{E}^{\mu\cdots\nu}_{\gamma\cdots\delta}[w]$. Thus, for example, the tractor-D maps $\mathcal{E}[w]$ into completely skew valence p cotwistors of weight $w-1$, $\mathcal{E}_{[\alpha\cdots\beta]}[w-1]$. Let us call $\mathcal{E}_{[\alpha\cdots\beta]}$ the *cotractor bundle*. We will use upper case Greek indices to indicate the abstract indices of the cotractor bundle and its tensor products and so forth. Thus, for example, we write

$$\mathcal{E}_{\Theta} = \mathcal{E}_{\underbrace{[\alpha\cdots\beta]}_{p}},$$

and similarly $\mathcal{E}^{\Theta}$ for the dual *tractor bundle*.

The tractors and cotractors come from G-modules, so they are special cases of what we have called twistors above. In contrast to the fundamental twistors, their filtrations are of length $p+1$. We shall see in a moment, that we recover the tractors of the conformal Riemmanian geometries in the case $p = 2 = q$.

The $p = 2$ case: In this case the tractor bundle is $\wedge^2\mathcal{E}^{\alpha}$ and we have

$$\mathcal{E}^{[\alpha\beta]} = \mathcal{E}^{[AB]} + \mathcal{E}^{AB'} + \mathcal{E}^{[A'B']}.$$

Using the canonical volume form $\epsilon^{A'B'}$, this may be rewritten as

$$\mathcal{E}^{\Theta} = \mathcal{E}^{[AB]} + \mathcal{E}^{A}_{B'}[-1] + \mathcal{E}[-1].$$

In this tractor notation $X^{\rho\sigma}$ of (30) is the canonical weight one tractor giving the injection $\mathcal{E}[-1] \to \mathcal{E}^{\Theta}$ by $f \mapsto fX^{\Theta}$. In any choice of scale, we have

$$X^{\Theta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In the cases $q > 2$ no such simplification is available for the analogous canonical object Y^{AB}_{Θ} which describes the injecting part of the cotractors. Nevertheless it is worthwhile noting that, in each choice of scale, it is given $Y^{CD}_{\Theta} = (\delta^{C}_{[A}\delta^{D}_{B]},\ 0\ ,\ 0)$. (Here, as above, we write the injecting part on the left here for consistency with [2].)

Observe that in the $q = 2$ case we completely recover the tractors from [2]. In particular, $h_{\Theta\Lambda} = h_{\alpha\beta\gamma\delta}$ is precisely the tractor metric described in [2,14] and in this case $Y^{AB}_{\Theta} = X_{\Theta}\epsilon^{AB}$ where $X_{\Theta} := h_{\Theta\Lambda}X^{\Lambda}$.

Using the expansions of $\mathsf{D}^{\rho\sigma}_{\alpha\beta}f$ as in Section 4, or otherwise it is easy to describe explicitly the form of the tractor-D operator for the $p = 2$ structures. Let $\tilde{D}_{\Theta}$ be the differential operator which, in a given choice of scale, may be written $\tilde{D}_{\Theta}f = (0, \nabla_a f,\ wf)$ for f

any weight w twistor (remember that tractor bundles may be thought of as twistor bundles). This is not itself invariant but in terms of this the invariant operator D_Θ is given

$$D_\Theta f = (w+1)\tilde{D}_\Theta f + Y_\Theta^{AB}\Box_{AB} f$$

where, again, f is any tractor of weight w and $\Box_{AB}$ is the operator given by the formula (21) (of course $\Box_{AB}$ is only invariant when $w = -1$). It is easily verified that when $q = 2$ this agrees with the usual formula for the tractor-D operator (apart from an overall factor of 2, – compare for example the formulae in [15]).

Salamon's complex. A subcomplex in the De Rham complex on a quaternionic manifold M was discussed in [22]. It is just a matter of observation that such a subcomplex appears for all torsion-free AG-structures with $2 = p < q$. This occurs in the BGG resolution of the sheaf of constant functions, see Fig. 1 in Appendix A describing the special case $p = 2$, $q = 4$. Observe that in that case we can obtain a longer complex if we bypass the bundle in the vertex of the triangle in Fig. 1 via the second order operator indicated by the vertical arrow and then continue on the border of the triangle down to the top degree forms. All this follows immediately from the fact that the whole diagram, viewed row after row is a genuine resolution. In fact it is easily verified that this result is typical and there is an analogous lengthening of Salamon's subcomplex for all torsion-free AG-structures with $2 = p < q$. Using any scale, all the first order operators are always given by the appropriate projections of the exterior derivatives expressed in terms of covariant derivatives. The 'bridging' second order operator is given in general by

$$u_{[A\cdots C]}^{(A'\cdots C')} \mapsto \nabla_S^{S'}\nabla_R^{(R'}u_{A\ldots C}^{A'\cdots C')}\epsilon^{RA\cdots C}\epsilon_{S'R'}.$$

Acknowledgements

Discussions with Andreas Čap and Michael Eastwood were important. Experimenting with Brian Boe's computer program for computing the classification lists of homomorphisms between generalized Verma modules has been also very useful (cf. [4]). Essential parts of the research were done during the second author's stays at the University of Adelaide and QUT in Brisbane, and the first author's stay at Masaryk University in Brno. Some writing was also done during the authors' visit at Erwin Schrödinger Institute in Vienna.

Appendix A. The Cartan connections of AG-structures

The AG-structures are specific examples of the so-called Cartan geometries. In general, we have in mind certain deformations of homogeneous spaces G/P and the main defining objects are the Cartan connections on principal P-bundles $\mathcal{G}$. See [23] for a complete exposition of the general ideas.

The aim of this appendix is to apply the general theory to the AG-structures and to provide some background for the main development in this article.

The *Cartan connections* are right invariant forms in $\Omega^1(\mathcal{G}, \mathfrak{g})$ which reproduce the fundamental vector fields for the principal action of P, and provide isomorphisms $T_u\mathcal{G} \to \mathfrak{g}$ for all $u \in \mathcal{G}$. The homogeneous cases are then just the left Maurer–Cartan forms ω on $G \to G/P$. An important class among such structures is characterized by two requirements: the semisimplicity of G, and the existence of the grading of the Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, $k \in \mathbb{Z}$, with $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ (the so-called $|k|$-graded Lie algebras). The Lie subgroup P corresponds then to the subalgebra $\mathfrak{p}$ and it is always a semi-direct product of its reductive part G_0 (with Lie algebra $\mathfrak{g}_0$) and the nilpotent exponential image P_+ of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. In all these cases, the corresponding geometries are defined in a way similar to classical G-structures and the canonical bundles $\mathcal{G}$, together with the canonical Cartan connections, are constructed from such data. The obstruction against the local equivalence to the homogeneous spaces is given by the curvature of the Cartan connection, the two-form $\kappa \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by structure equation

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \kappa.$$

By definition, the curvature κ is a horizontal 2-form and the presence of the absolute parallelism ω itself enables us to view κ as the P-equivariant function

$$\kappa : \mathcal{G} \to \mathfrak{g}_-^* \wedge \mathfrak{g}_-^* \otimes \mathfrak{g},$$

where $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is identified with $\mathfrak{g}/\mathfrak{p}$. In our case, the algebra is $|1|$-graded and so the curvature splits into components κ_{-1} (the *torsion part*), κ_0 (the *Weyl part*) and κ_1.

The canonical Cartan connections are normalized to have co-closed curvatures κ, i.e. $\partial^* \circ \kappa = 0$, with respect to the adjoint to the Lie algebra cohomology differential ∂. Such Cartan connections are constructed (including the bundle $\mathcal{G}$) from simple geometric data on the underlying manifold, see, e.g. [5] or [25] for explicit constructions in the most general situations. A very detailed exposition is also available in [27].

The best known examples are the conformal Riemannian structures and the projective geometries, and all $|1|$-graded cases behave very much similar to them, cf. [3,6,7]. The name *AG-structures* refers in general to all $|1|$-graded cases where the complexification of $\mathfrak{g}$ is $\mathfrak{sl}(p+q, \mathbb{C})$. In fact, there are only four relevant series of geometric structures, cf. [16,17]:

(1) $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$ and $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(q, \mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{g}_1 = \mathbb{C}^{q*} \otimes_{\mathbb{C}} \mathbb{C}^p$,

(2) $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbb{R}) \oplus \mathfrak{sl}(q, \mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}_1 = \mathbb{R}^{q*} \otimes_{\mathbb{R}} \mathbb{R}^p$,

(3) $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{H})$ and $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbb{H}) \oplus \mathfrak{sl}(q, \mathbb{H}) \oplus \mathbb{R}$, $\mathfrak{g}_1 = \mathbb{H}^{q*} \otimes_{\mathbb{H}} \mathbb{H}^p$

(4) $\mathfrak{g} = \mathfrak{su}(p, p)$ and $\mathfrak{g}_0 = \mathfrak{csl}(p, \mathbb{C})$, $\mathfrak{g}_1 = (\mathfrak{su}(p))^*$.

A general calculus for differential geometry of all $|1|$-graded geometries was developed in [6], see also [24]. We are going to review briefly some of the general features of this and present explicit formulae for the AG-structures.

The intuitive explanation of what the geometries look like is as follows: In each case the tangent space is identified with the negative part $\mathfrak{g}_{-1}$ of the Lie algebra $\mathfrak{g}$, as a G_0-module. The most natural choice of the Lie group G_0 with Lie algebra $\mathfrak{g}_0$ is the adjoint group of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$. This choice leads to a sort of minimal data and in all $|1|$-graded cases this amounts to a classical G-structure, i.e. a reduction of the general linear frame bundle to the structure group G_0. The Cartan bundle $\mathcal{G}$ and the Cartan connection ω are then built out of these data.

In the case of $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$ the structure group described above is a quotient $\tilde{G}_0$ of $G_0 = S(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$, where $G_0 \to \tilde{G}_0$ is a $(p+q)$-fold covering. Thus, it is convenient to work with the whole G_0 instead which, of course, adds some global structure to our geometries. It does not play any important role locally though. (In fact, the situation is similar to the spin structures on conformal Riemannian structures, cf. the case $p = q = 2$.) In this paper, we are always assuming that this additional structure is given. Then the G_0 structure yields an identification of the tangent space of the complex manifold M with the tensor product of two auxiliary (complex) vector bundles $TM = \mathcal{E}^A \otimes \mathcal{E}_{A'}$, together with the fixed isomorphism of their top degree exterior products, cf. [1].

The real split form $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$ leads exactly to the same description, except we replace complex manifolds and vector bundles by the real ones, and the reductive group $G_0 = S(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R}))$ equals the minimal structure group $\tilde{G}_0$ if $p+q$ is odd, while $G_0 \to \tilde{G}_0$ is a 2-fold covering if $p+q$ is even.

The other two real forms are more interesting and quite different, but we can still include them into the above framework if we deal with the complex P-modules and the complexified tangent bundle $TM \times_{\mathbb{R}} \mathbb{C}$. Thus we are using the same abstract index formalism for all these structures, but we have to keep in mind that it is, with p and q even, the quaternionic form $\mathfrak{sl}(p/2 + q/2, \mathbb{H})$ which corresponds then to the discussion of the cases with $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$. This is also compatible with the developments in [1,22].

Let $\mathcal{G}$ be the Cartan bundle equipped with the normal connection ω. The quotient bundle $\mathcal{G}_0 = \mathcal{G}/\exp \mathfrak{g}_1$ is a principal fibre bundle with structure group G_0. Moreover, there is the family of global G_0-equivariant sections $\sigma : \mathcal{G}_0 \to \mathcal{G}$ parametrized by 1-forms on M and each such section σ induces the linear connection $\gamma^\sigma := \sigma^*\omega_0$ on M (viewed as a principal connection on $\mathcal{G}_0$). The latter connection, together with the soldering form $\theta := \sigma^*\omega_{-1}$ on $\mathcal{G}_0$, forms a Cartan connection in $\Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$, and there is the σ-related Cartan connection $\omega^\sigma \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The $\mathfrak{g}_1$-component of the latter connection ω^σ has to vanish on $T\sigma(T\mathcal{G}_0)$, while the $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$-components of ω and ω^σ coincide. This implies that these Cartan connections are related by

$$\omega^\sigma = \omega - \mathsf{P} \circ \omega_{-1}, \tag{A.1}$$

where $\mathsf{P} : \mathcal{G} \to \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1$ enjoys the equivariance properties of a 2-tensor on M. The latter tensor is called the *Rho-tensor* defined by the choice of σ. The whole torsion part of the curvature κ of the Cartan connection ω is constant on the fibres of $\mathcal{G}$ and provides exactly the torsion shared by all connections γ^σ.

The absolute parallelism ω defines the *horizontal vector fields* $\omega^{-1}(X)$ for all $X \in \mathfrak{g}_{-1}$. Now, for each P-module V we have the natural vector bundles $\mathcal{V}$ associated to $\mathcal{G}$ and their sections may be viewed as P-equivariant functions $s : \mathcal{G} \to V$. The *invariant differential* ∇^ω given by the Cartan connection ω is then the obvious differentiation in the directions of the horizontal vector fields:

$$\nabla^\omega : C^\infty(\mathcal{G}, V) \to C^\infty(\mathcal{G}, \mathfrak{g}_{-1}^* \otimes V), \qquad \nabla^\omega_X s(u) = \omega^{-1}(X)(u) \cdot s.$$

In particular, in terms of these invariant derivatives the Ricci and Bianchi identities have the form

$$(\nabla^{\omega}_X \circ \nabla^{\omega}_Y - \nabla^{\omega}_Y \circ \nabla^{\omega}_X)s = \lambda(\kappa_{\mathfrak{p}}(X, Y)) \circ s - \nabla^{\omega}_{\kappa_{-1}(X,Y)}s, \tag{A.2}$$

$$\sum_{\text{cycl}}([\kappa(X, Y), Z] - \kappa(\kappa_{-}(X, Y), Z) - \nabla^{\omega}_Z \kappa(X, Y)) = 0, \tag{A.3}$$

where λ means the representation of $\mathfrak{p}$ in $\mathfrak{gl}(V)$, $X, Y, Z \in \mathfrak{g}_{-1}$.

For irreducible P-modules V (and all those with trivial actions of $\mathfrak{g}_1$) we can easily compare the invariant differentials with the covariant derivatives with respect to any section σ. We obtain

$$(\nabla^{\omega}_X - \nabla^{\gamma^{\sigma}}_X)s(u) = \lambda([X, \tau(u)]) \circ s(u), \tag{A.4}$$

where $\tau : \mathcal{G} \to \mathfrak{g}_1$ is defined by $u = \sigma(p(u)) \exp \tau(u)$ and it measures the distance of u from the image $\sigma(\mathcal{G}_0)$ in $\mathcal{G}$. Consequently, the transformation of the first derivatives in terms of the change of the scale is

$$\nabla^{\hat{\gamma}}_X s = \nabla^{\gamma}_X s + \lambda([X, \Upsilon]) \circ s. \tag{A.5}$$

Let us work out this formula in our abstract index formalism. First of all we need formulae for brackets of elements in $\mathfrak{g}$. We shall write typical elements $X \in \mathfrak{g}_{-1}$, $Y \in \mathfrak{g}_0$, and $Z \in \mathfrak{g}_1$ as

$$X = v^A_{A'}, \quad Y = (u^{A'}_{B'}\delta^B_A + u^B_A \delta^{A'}_{B'}), \quad Z = w^{A'}_A.$$

Notice that the convention for $\mathfrak{g}_0$ follows the obvious embedding of $\mathfrak{g}_0$ into the endomorphisms $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$. In this notation, the brackets in the matrix Lie algebra $\mathfrak{g}$ can be expressed by

$$[Y, X] = -u^{B'}_{A'}v^A_{B'} + u^A_B v^B_{A'},$$
$$[Y, Z] = u^{A'}_{B'}w^{B'}_A - u^B_A w^{A'}_B,$$
$$[X, Z] = -w^{A'}_C v^C_{B'}\delta^B_A + v^B_{C'} w^{C'}_A \delta^{A'}_{B'}.$$

Now, the expression $\mathfrak{g}_{-1} \ni X \mapsto [X, \Upsilon] \in \mathfrak{g}_0$ with $X = v^A_{A'}$ and $\Upsilon = \Upsilon^{A'}_B \in \mathfrak{g}_1$, appearing in (A.4), can be understood as

$$v^A_{A'} \mapsto (-\Upsilon^{D'}_A \delta^{A'}_{C'}\delta^C_D + \Upsilon^{A'}_C \delta^D_A \delta^{C'}_{D'})v^A_{A'}.$$

Thus in order to obtain the formula (A.4) we have to act by the element $(-\Upsilon^{D'}_A \delta^{A'}_{C'}\delta^C_D + \Upsilon^{A'}_C \delta^D_A \delta^{C'}_{D'})$, viewed as a $\mathfrak{g}_0$-valued 1-form with free indices ${}^{A'}_{A}$, composed with the representation λ. This yields immediately the formulae in (4).

The Cartan connection ω induces a connection on all natural bundles coming from G-modules and the corresponding covariant derivative ∇ is compared to the invariant derivative (and covariant derivatives with respect to the linear connections γ^{σ}) by the formula

$$\nabla_X s = \nabla^{\omega}_X s + \lambda(X) \circ s = \nabla^{\gamma^{\sigma}}_X s - \lambda(\mathsf{P} \cdot X) \circ s + \lambda(X) \circ s.$$

Again, the explicit formulae (14), (15) follow immediately.

The transformation rule for P under the change given by Υ is then

$$\hat{\mathsf{P}} \cdot X = \mathsf{P} \cdot X - \nabla_X \Upsilon - \tfrac{1}{2}[\Upsilon, [\Upsilon, X]]. \tag{A.6}$$

In our index formalism this yields exactly (10).

Next, let us discuss the normalizing conditions on the curvatures. The general formula for the Lie algebra cohomology codifferential ∂^* (applied to 2-forms in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes W$ for a $\mathfrak{g}$-module W) reads

$$\partial^*(Z_1 \wedge Z_2 \otimes v) = -Z_2 \otimes Z_1 \cdot v + Z_1 \otimes Z_2 \cdot v$$

and so its evaluation on the torsion $T_{ab}{}^c = F_{ABC'}^{A'B'C} + \tilde{F}_{ABC'}^{A'B'C}$ where $F_{ABC'}^{A'B'C} = F_{(AB)C'}^{[A'B']C}$ and $\tilde{F}_{ABC'}^{A'B'C} = \tilde{F}_{[AB]C'}^{(A'B')C}$ yields

$$\partial^*(T_{ab}{}^c) = 2(-T_{ABD'}^{D'B'C}\delta_{C'}^{A'} + T_{DBC'}^{A'B'D}\delta_A^C).$$

The vanishing of this expression is equivalent to the vanishing of all traces of the objects $F_{(AB)C'}^{[A'B']C}$, $\tilde{F}_{[AB]C'}^{(A'B')C}$.

Similarly, the evaluation of the codifferential on the $\mathfrak{g}_0$-component $U_{ABD'}^{A'B'C'}\delta_C^D + \tilde{U}_{ABC}^{A'B'D}\delta_{D'}^{C'}$ of the curvature κ yields

$$\partial^*(\kappa_0) = 2(-U_{ABD'}^{D'B'A'} + \tilde{U}_{DBA}^{A'B'D})$$

and the condition $\partial^*\kappa_0 = 0$ is equivalent to the vanishing of the two contractions on the right-hand side.

By the construction and the general theory, the curvatures κ^σ of the Cartan connections ω^σ are σ-related to the sum of torsions and curvatures of the induced linear connections γ^σ on $\mathcal{G}_0$. At the same time, the relation between κ^σ and κ is

$$\begin{aligned}(\kappa^\sigma - \kappa)(u)(X, Y) = {} & \partial\mathsf{P}(u)(X, Y) + \nabla_X^\omega \mathsf{P}(u) \cdot Y \\ & -\nabla_Y^\omega \mathsf{P}(u) \cdot X + \mathsf{P}(u) \circ \kappa_{-1}^\sigma(u)(X, Y).\end{aligned} \tag{A.7}$$

Our description of the curvature of the twistor connection, see (16), is an immediate consequence of this formula. Furthermore, the $\mathfrak{g}_0$-component of this expression yields exactly our formula (6).

The general theory also shows that the whole curvature vanishes if and only if its harmonic part vanishes and this in turn can be computed explicitly by Kostant's version of Bott–Borel–Weil theorem. In our case this means that the whole curvature is determined by the two components F and $\tilde{F}$ of the torsion if $2 < p \leq q$. In the case $p = 2 < q$ only one of the torsions survives, $\tilde{F}$, and there appears another invariant component of $\tilde{U}_{ABC}^{A'B'D}$, namely the completely trace-free part of $\tilde{U}_{(ABC)}^{[A'B']D}$. Let us also notice, that if the torsion happens to vanish, then the latter component of the Weyl curvature is constant along the fibres of $\mathcal{G} \to \mathcal{G}_0$ and there is no other non-zero component in the Weyl part of the curvature. Moreover, in this case, the $\mathfrak{g}_0$-component of the Bianchi identity (A.3) yields for all $X, Y, Z \in \mathfrak{g}_{-1}$

$$-\partial\kappa_1(X, Y, Z) = \sum_{\text{cycl}} \nabla_Z \kappa_0(X, Y).$$

An easy computation reveals that the right-hand side is in the kernel of ∂. Because there is no cohomology in that place, the latter equation has a unique solution for κ_1 in terms of the derivatives of the only non-zero component in κ_0, i.e. of the Weyl spinor $\tilde{U}^{[A'B']}_{(ABC)}$.

The invariant linear operators between natural bundles over locally flat AG-structures are in bijective correspondence with the homomorphisms of generalized Verma modules. Thus they are well known from representation theory. In particular, all cases with the so-called regular infinitesimal character are obtained by the translation of the standard De Rham resolution of the sheaf of constant functions. This is the source of the celebrated Bernstein–Gelfand–Gelfand resolutions (briefly BGG resolutions).

The complete BGG resolution of $\mathcal{E}$ in the special case $p = 2$, $q = 4$ (i.e. the lowest dimensional interesting quaternionic geometry) is shown in Fig. 1. The long arrows on the left-hand side denote the non-standard operators. One of the aims of our development is to provide tools for extending such operators to curved geometries. In fact, there are several methods available, but mostly they fail if applied to non-standard operators. Also the arrows along the side of the triangle joining $\mathcal{E}$ and $(\square\square\square\square\,\mathcal{E}^{A'})[-1]$ are worth mentioning. Namely, they form Salamon's subcomplex on quaternionic structures.

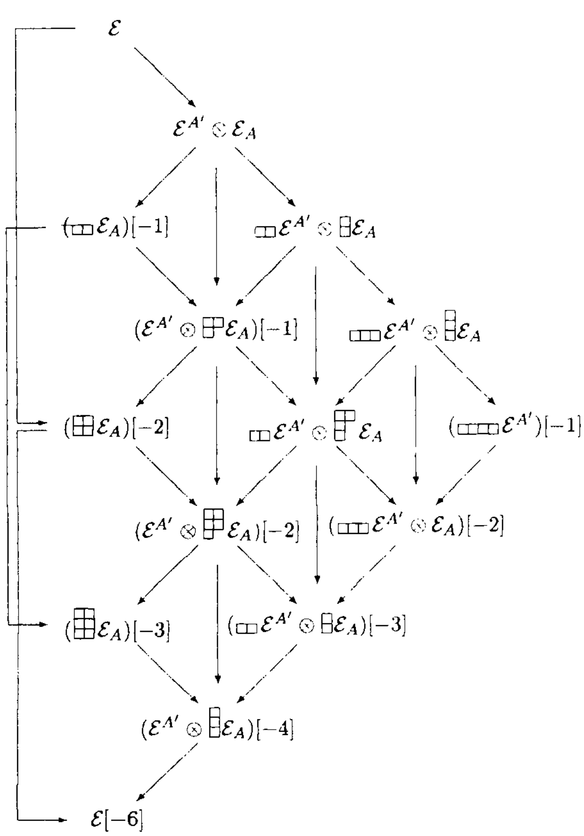


Fig. 1.

Appendix B. Normal forms for AG-structures

Given a choice of scale one has a connection ∇^ξ on M and for each point $q \in M$ one can define normal coordinates x^i in a neighbourhood of q. Up to a general linear transformation, such coordinates may be characterized by the conditions that (1), $x^i(q) = 0$, that (2) the vectors $\partial/\partial x^i|_q$ give a G_0-frame at q and that (3) the coefficients $\Gamma_{(\xi)}$ of ∇^ξ, in these coordinates, satisfy

$$\Gamma^i_{jk} x^j x^k = 0 \tag{B.1}$$

in the neighbourhood where the coordinates are defined. Note that

$$\Gamma^i_{(jk)}(q) = 0,$$

and so, at q,

$$\Gamma^i_{jk} = T_{jk}{}^i.$$

Similarly differentiating (B.1) with respect to the normal coordinates and evaluating at q we obtain that $\partial_{(i}\Gamma^i_{jk)}(q) = 0$. It follows easily that, at $q \in M$,

$$\partial_k \Gamma^l_{ij} = 2R_{k(ij)}{}^l + \tfrac{1}{6}(3\partial_k T_{ij}{}^l + 2\partial_{(i} T_{j)k}{}^l) + \tfrac{1}{2} T_{m[k}{}^l T_{j]i}{}^m + \tfrac{1}{2} T_{m[k}{}^l T_{i]j}{}^m.$$

The partial derivatives on the right-hand side, of the above, may be replaced with covariant derivatives at the expense of adding more terms quadratic in the (undifferentiated) torsion. By an obvious inductive argument one can easily continue in this manner and recover the following established result.

Proposition B.1. *In terms of the normal coordinates for ∇^ξ, based at $q \in M$, the coefficients of the Taylor series of the $\Gamma_{(\xi)}$ are given by polynomial expressions involving the components of the ∇^ξ covariant derivatives of the curvature and torsion of ∇^ξ.*

Clearly for any choice of scale ξ and $q \in M$ we can find a G_0 family of such normal coordinates.

Fix a choice of scale and normal coordinates in a neighbourhood of $q \in M$. Let u^a be a tangent vector at q and u^i its components in the normal coordinates. Suppose this is extended to a section of the tangent bundle in a neighbourhood of q by parallel transporting u^a along the geodesics through q. Then $x^i \nabla^\xi_i u^a = 0$ and it is an elementary exercise using this to show that the coefficients of Taylor series of u^a, about q and in the normal coordinates, are given by polynomials in the components $u^i(q)$ and the

$$\Gamma^i_{ij},\underbrace{{}_{k\cdots l}}_{t}(q), \tag{B.2}$$

for $t = 0, 1, \ldots$ (Here $\Gamma^i_{ij},{}_{k\cdots l} := \partial_l \cdots \partial_k \Gamma^i_{ij}$ and the polynomials just described are homogeneous of degree 1 in the components $u^i(q)$.) It follows that the coefficients of the Taylor series of the *normal G_0-frame*, corresponding to the normal coordinates, are polynomial in coefficients Γ^i_{jk} and their normal coordinate derivatives at q. This frame is obtained by parallel transporting the frame $\partial/\partial x^i|_q$ along the geodesics through q. It follows easily that the coefficients of the connection ∇^ξ in this normal G_0-frame have normal coordinate Taylor series with coefficients also polynomial in the variables (B.2). We have a corresponding result for *normal spin frames*. These are constructed as follows. Choose spin frames for $\mathcal{E}^A(q)$ and $\mathcal{E}_{A'}(q)$ consistent with the G_0-frame $\partial/\partial x^i|_q$ at q given by the normal coordinates. Now using the spin connections ∇^ξ parallel transport these frames along the geodesics through q. This determines normal G_0-frames for $\mathcal{E}^A(q)$ and $\mathcal{E}_{A'}$ in a neighbourhood of q. Let Γ^A_{Bi} and $\Gamma^{A'}_{B'i}$ be the coefficients of the spin connections with respect to these frames, where the index i refers to the normal coordinates (and the indices A, B, A', B' here are concrete indices). These coefficients are linear combinations of the coefficients of the normal G_0-frame. Thus, with the proposition above we have the following.

Proposition B.2. *Given a scale ξ, and normal coordinates at x^i, based at $q \in M$, let Γ^A_{Bi} and $\Gamma^{A'}_{B'i}$ be the coefficients of the spin connections with respect to the normal spin frame. The coefficients of the Taylor series of these functions are given by polynomial expressions involving the components of the ∇^ξ covariant derivatives of the curvature and torsion of ∇^ξ.*

Given the point $q \in M$ we can also normalize the scale, at least formally. Using Eq. (10), and by considering formal power series, it is easily verified that one can choose a scale so that

$$\underbrace{S_{(a\cdots ef)}}_{s}(q) = 0 \tag{B.3}$$

for $s = 2, 3, \ldots, r$ for any given $2 \le r \in \mathbb{N}$. Let us suppose that we have chosen and fixed r so that it is sufficiently large for our calculations and denote this preferred scale ξ_q.

Remark B.3. *In fact it is clear from the form of* (10) *that the condition* (B.3) *leaves the 1-jet at q of ξ_q completely free. Thus beginning with any scale ξ and an arbitrary point $q \in M$, one can achieve a normal scale based at q, ξ_q by a transformation $\xi_q = \Omega\xi$ where $\Upsilon_a(q) = 0$.*

Although we will not use it directly here it is worth observing that, in this scale the Taylor series of Proposition B.1 simplifies somewhat. Recall the decomposition (6) of the curvature. It is clear that the jets of the curvature $R^{(\xi)}{}_{ab}{}^c{}_d$ are given linearly by the jets of the tensor $U^{(\xi)}{}_{ab}{}^c{}_d$ and the jets of the Rho-tensor $\mathsf{P}^{(\xi)}_{ab}$. Considering various Young projectors acting on $\nabla_a\nabla_b\cdots\nabla_d\mathsf{P}^{(\xi_p)}_{ef}$ one easily concludes that, at $q \in M$, this tensor is determined by $\nabla_a\nabla_b\cdots\nabla_d\mathsf{P}^{(\xi_p)}_{[ef]}$, $\nabla_a\nabla_b\cdots\nabla_{[d}\mathsf{P}^{(\xi_p)}_{e]f}$ and lower order terms. But, by (9) $\mathsf{P}^{(\xi)}_{[ef]}$ is given by a linear formula in terms of a ∇^ξ derivative of the torsion. Thus we obtain the following simplification to the above proposition.

Proposition B.4. *Let $q \in M$ and ξ_p be a scale such that* (B.3) *is satisfied. Let x^i be normal coordinates for ∇^{ξ_p} based at q. Then, in terms of these coordinates, the coefficients* *of the Taylor series of the* $\Gamma_{(\xi_p)}$ *(to order* $r+1$*) are given by polynomial expressions in the components of the covariant derivatives of the torsion* $T_{ab}^{(\xi_p)\,c}$ *of* ∇^{ξ_p} *and the components of the covariant derivatives of the tensors* $U_{ab}^{(\xi_p)c}{}_d$ *and* $\nabla_{[a}\mathsf{P}^{(\xi_p)}_{b]c}$.

Appendix C. Composition series

In the following discussion we will review several notions and terms for representations of a group H. We have, for the most part, not said anything about the nature of this group since an explicit description of the group is not required for most of the results here. Of course for application of these results to the other parts of this article one may take H to be a parabolic P in one of real Lie groups G as discussed in the introduction. We would also like to point out that in this case the terms introduced (such as "composition series" and "injecting part" etc.) can be adapted in an obvious way to the natural bundles that P induces and indeed to differential operators that take values in such natural bundles. Throughout the article we have used this observation without other mention.

Suppose V is an H-module for some group H. Let W be an H-submodule of V then we have an exact sequence

$$0 \to W \to V \to U \to 0,$$

where U is the required quotient. Following Buchdahl (see also [2]) it is often convenient to express this as a composition series in the following schematic manner:

$$V = U + W.$$

Suppose now that V is any non-trivial finite dimensional module for the group H. We construct a composition series of V as follows. Let V_1^1 be an irreducible submodule of V. If there is a non-trivial submodule of V in a complement to V_1^1 then there is at least one irreducible one which may denote V_1^2. Continuing in this manner suppose that $\{V_1^1, V_1^2, \dots, V_1^{m_1}\}$ is a *maximal* set of such submodules, meaning that there are no non-trivial submodules of V in a complement to $V_1 := \oplus_{i=1}^{m_1} V_1^i$. We call V_1 the first *composition factor* of V, while the irreducible submodules V_1^i ($i \in \{1, \dots, m_1\}$) in this, will be described as *injecting parts* of V.

Now let $U_2 := V/V_1$. Then U_2 is an H-module and so we may similarly choose a set of irreducible submodules of this, V_2^i, $i = 1, \dots, m_2$, such that this set is maximal in U_2. We write V_2 for the first composition factor of U_2, that is $V_2 = \oplus_{i=1}^{m_2} V_2^i$.

Now we may consider $U_3 := U_2/V_2$ and seek a maximal set of irreducible submodules of this (which we denote V_3^i, $i = 1, \dots, m_3$) and so on. Note that at any stage $V_t = 0$ if and only if $U_t = 0$. Since the U_{t+j}, for $j \geq 1$, are quotients of U_t it follows that $V_t = 0$ implies $V_{t+j} = 0$ for all $j \geq 1$. In fact since V is assumed finite dimensional it is clear that there exists some positive integer r such that $V_{r+1} = 0$ while $V_r \neq 0$. With that determined the *composition series* of V is given,

$$V = \left(\bigoplus_{i=1}^{m_r} V_r^i\right) + \left(\bigoplus_{i=1}^{m_{r-1}} V_{r-1}^i\right) + \cdots + \left(\bigoplus_{i=1}^{m_1} V_1^i\right). \tag{C.1}$$

We describe $V_k = \oplus_{i=1}^{m_k} V_k^i$ as the kth *composition factor* of V. The V_r^i $(i \in \{1, \ldots, m_r\})$ will be called the *projecting parts* of V. (It is usual to describe $V_r + V_{r-1} + \cdots + V_1$ as the composition series for V. For our purposes it is convenient to choose a decomposition of the composition factors as indicated.)

We have the following results.

Theorem C.1. *Suppose an H-module V has a composition series as in* (C.1). *Then for S an H-submodule of V we have*

$$S \cap V_1 = 0 \Leftrightarrow S = 0.$$

Proof. The implication $\Leftarrow$ is clear. Suppose now S is an *irreducible* H-submodule such that $S \cap V_1 = 0$. Then $S = 0$ since $\{V_1^1, \ldots, V_1^{m_1}\}$ is a maximal set of irreducible submodules of V. Now suppose S is any H-submodule such that $S \cap V_1 = 0$. Then an irreducible H-submodule S' of S is an irreducible H-submodule of V such that $S' \cap V_1 = 0$. Thus by the established result $S' = 0$. Thus S has no non-trivial irreducible submodules and so $S = 0$ as claimed. □

The following indicates that a composition series is unique up to some possible choice for the splitting of each part into irreducibles.

Corollary C.2. *Suppose an H-module V has a composition series as in* (C.1) *and also a composition series*

$$V = \left(\bigoplus_{i=1}^{\tilde{m}_{\tilde{r}}} \tilde{V}_{\tilde{r}}^i\right) + \left(\bigoplus_{i=1}^{\tilde{m}_{\tilde{r}-1}} \tilde{V}_{\tilde{r}-1}^i\right) + \cdots + \left(\bigoplus_{i=1}^{\tilde{m}_1} \tilde{V}_1^i\right)$$

then $\tilde{r} = r$, $\tilde{m}_1 = m_1, \ldots, \tilde{m}_r = m_r$ and $V_1 = \tilde{V}_1 := \oplus_{i=1}^{m_1} \tilde{V}_1^i, \ldots, V_r = \tilde{V}_r := \oplus_{i=1}^{m_r} \tilde{V}_r^i$. Furthermore in each composition factor V_k one can arrange the numbering of the V_k^i so that for each $i \in \{1, \ldots, m_k\}$ $V_k^i \cong \tilde{V}_k^i$. If for any i the module V_k^i occurs with multiplicity 1 in V_k then we get $V_k^i = \tilde{V}_k^i$.

Proof. The first part of this is immediate by repeated application of the theorem while the last part follows from Schur's lemma. □

From this in turn we get the following corollary.

Corollary C.3. *Let V be an H-module with composition series as in* (C.1). *If S is an H-submodule of V then S has a composition series*

$$S = \left(\bigoplus_{i=1}^{l_{r_s}} S_{r_s}^i\right) + \left(\bigoplus_{i=1}^{l_{(r_s-1)}} S_1^i\right) + \cdots + \left(\bigoplus_{i=1}^{l_1} S_1^i\right),$$

where for each $k \in \{1, \ldots, r_s\}$ *and* $i \in \{1, \ldots, l_k\}$ *there is some* $j \in \{1, \ldots, m_k\}$ *such that*

$$S_k^i \cong V_k^j,$$

with equality if V_k^j *occurs with multiplicity 1 in* V_k.

Thus all homomorphisms between finite dimensional H-modules V and W are determined by the composition series for V and W, at least up to an isomorphism ambiguity due to the multiplicity of irreducible components in each part.

We are in particular interested in the composition series of P modules which are the restriction to P of irreducible G modules and also their P-submodules. Recall that P is a maximal parabolic in a group G which is a real form of the complex semi-simple groups $\mathrm{SL}(p+q, \mathbb{C})$. In this case some aspects of the composition series are rather easily described.

Let V_α be the dual to the standard representation of G. Then we have an exact sequence of P-modules

$$0 \to V_A \to V_\alpha \to V_{A'} \to 0. \tag{C.2}$$

Let Y_α^A be the canonical element of $V^A \otimes V_\alpha$ giving the injection $V_A \to V_\alpha$ and $X_{A'}^\alpha$ be the canonical element of $V_{A'} \otimes V^\alpha$ giving the surjection $V_\alpha \to V_{A'}$. (This notation is borrowed from the notation for the corresponding objects for bundles these modules induce.) Let us also write H_α for the image of V_A in V_α.

Now irreducible G-modules may be described by Young diagrams. Using notation as in Section 5 we may write for example

$$\mathbf{Y}(b) V_\alpha,$$

where $\mathbf{Y}(b)$ indicates a Young diagram with a total of b boxes. (We will suppose the height of this diagram is no greater than $p+q$ so this module is not trivial.) Elements of this module consist of vectors which carry b indices,

$$v_{\underbrace{\alpha \cdots \gamma}_{b}},$$

and a symmetry indicated by the Young diagram. Regard this now as a P-module by restriction and consider the subspace of vectors that have the property that they are X-*saturated*, that is they are annihilated upon contraction with $X_{A'}^\alpha$ on any index,

$$0 = X_{A'}^\alpha v_{\alpha\beta\cdots\gamma} = X_{A'}^\beta v_{\alpha\beta\cdots\gamma} = \cdots = X_{A'}^\gamma v_{\alpha\beta\cdots\gamma}.$$

The space of such vector clearly forms a P-submodule of $\mathbf{Y}(b) V_\alpha$. Considering each index in turn it is clear that it is a submodule of $\otimes^b H_\alpha$. Thus it is precisely the submodule

$$\mathbf{Y}(b) H_\alpha.$$

Of course this may be trivial but in any case

$$\mathbf{Y}(b) H_\alpha \cong \mathbf{Y}(b) V_A$$

and so it is irreducible. Thus if this is not zero then it gives the unique injecting part of $\mathbf{Y}(b)V_\alpha$ (which is therefore also the first composition factor). If the height of the diagram $\mathbf{Y}(b)$ is no greater than q then we are in this situation, that is $\mathbf{Y}(b)V_A \neq 0$, and we shall henceforth assume this is the case since it is sufficient for our purposes. The quotient

$$(\mathbf{Y}(b)V_\alpha)/V_1$$

may clearly be identified with the direct sum of the distinct images of $\mathbf{Y}(b)V_\alpha$ under the mapping given by contraction with one $X^\rho_{R'}$. Each of these distinct images carries a Young symmetry on its twistor indices (that is the greek indices) and, reasoning essentially as for the previous case, one sees that the irreducible parts of the second composition factor are the subspaces of images that are annihilated by any (further) contraction with $X^\sigma_{S'}$. One can clearly continue in this manner to determine the entire composition series. For the purposes of this article we only explicitly require an understanding of this to the level described. Let us just finally observe that given a choice of splitting of the sequence (C.2), or equivalently a choice of λ^β_B such that $\lambda^\beta_B Y^A_\beta = \delta^A_B$ it follows immediately from the observations here that we may describe these parts of the composition series as follows. The injecting part of $\mathbf{Y}(b)V_\alpha$ may be identified with the space of vectors $\lambda^\alpha_A\lambda^\beta_B\cdots\lambda^\gamma_C v_{\alpha\beta\cdots\gamma}$ for $v_{\alpha\beta\cdots\gamma}\in\mathbf{Y}(b)V_\alpha$. The second composition factor may similarly be identified with the vector space of objects consisting of vectors in $\mathbf{Y}(b)V_\alpha$ contracted into $(b-1)$ λ^ρ_R's and one $X^\rho_{R'}$. The corresponding result for induced bundles is used in Section 5.

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